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THE RECORD

American Institute of Actuaries

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THE RECORD

American
Institute of Actuaries

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Communications to the American Institute of Actuaries should be sent to the Secretary.

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Page 2, line 8: $\frac{D_{y+t}}{D_y} \cdot A_{x+t:x}$ should be $\frac{D_{y+t}}{D_y} \cdot a_{x+t:x}$

Page 13, line 10: $a_{36:n}$ should be $a_{36:n}$

Page 13: Table $n=10$, $t=15$: .92 should be .82

Page 17: Formula $(\frac{1}{2}+A_{x+t+\frac{1}{2}})$ should be $(\frac{1}{2}+a_{x+t+\frac{1}{2}})$

Page 20, middle: 88.36×14.222 should be 1188.36×14.222

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REMARKS BY PRESIDENT BUTTOLPH

The order of business calls for an address or remarks from the president. I am glad some choice is granted me. That there may be no misunderstanding on the subject, I wish to say that there will be a few remarks but no address.

The Institute has made progress during the past year—nothing startling, but that was not to be expected. It is gratifying that there has been some and that along lines which, if properly developed, will in time fully establish our reason for being.

In point of membership, there has been some gain. Several applications for membership have been received and acted upon favorably. But the justification for our existence is not going to be measured by the number of our members.

One most important feature of the Institute which was a part of the original plan is being developed steadily and wisely until already it is a thing of no inconsiderable value. I refer to the library. The librarian will doubtless give us during the course of the meeting information regarding this work. I will not attempt to anticipate his report but I do wish to emphasize the importance of this feature. The growth of the library doubtless will be slow but we already have a fine start and it is difficult to overestimate the value which will in time accrue to our members on this account.

Another feature not specifically included in our plan of organization was discussed by the Board of Governors at a meeting held in February and a committee was appointed to go fully into the matter of securing a recognition on the part of the schools and universities of the importance of introducing into their curricula at least a rudimentary course of insurance. I trust it will be your pleasure to have a report from this committee some time during this meeting as it seems to me that this is a matter in which we should all be deeply interested and which each one should help along to the fullest extent possible.

Thus far the Institute has little of history to look back upon. It is therefore natural that our thoughts should turn to the future. There has scarcely been time to work out anything in the shape of a policy for this future. What should it be? I confess, my

own ideas all seem to take the form of questions—questions which I am not yet ready to attempt to answer. While some of you have already won your individual spurs in a long useful service to the business which we represent, yet the Institute has not. It did not spring Minerva-like, fully matured and fully armed, from the brain of a Jupiter. It therefore seems to me that we can all, with advantage to ourselves and the society we have founded, give serious thought to the question of what is to be the policy for the future of this body. I myself have had some doubts as to whether or not we were altogether wise in thus early in our existence providing for admission by examination. Should our standard either now or at any time in the future be such a one as can be measured by an examination? Or should we, while maintaining as high a standard as any similar body so far as honesty of purpose and method are concerned, yet build on a broader foundation than would be consistent with such a requirement?

Whatever may be your opinion now upon this and kindred subjects, or whatever opinion you may in the future form, I think all will agree with me in saying that the future of the organization must depend upon work of the individual members. My faith in the ultimate success of this, our undertaking, is great. I believe that every member who makes proper use of his membership will receive great benefits from that membership and therefore I urge that each of us look upon himself as a future, if not a present debtor to the Institute. Let us, in this formative period of our organization, not use our membership for individual exploitation but do everything in our power to increase her prestige and her influence. If this is to be accomplished, these meetings must be made productive of something really worth while. They are and always will be the very life of the organization. Each member should stand ready to give of the very best that is in him to make them an inspiration of greater endeavor on the part of all who attend.

It was to me a matter of most sincere regret that the Board of Governors decided that it would be unwise to hold a meeting last fall, although I was convinced of the wisdom of that decision. When the matter was discussed by you a year ago, the question was left to the board to decide, but the decision must really be made by the whole body. The board can call a fall meeting but you alone can make it a success. A selfish motive should not be necessary, yet there can be no doubt but that individual effort

expended primarily for the sake of making these meetings a success, will bring material rewards to the individual himself in more than one way.

I hope the inauguration of successful fall meetings is near at hand.

ON THE FORMATION OF COMMUTATION COLUMNS FOR TOTAL AND PERMANENT DISABILITY BENEFITS

By LUCIUS McADAM

The introduction by many life insurance companies of the Total and Permanent Disability Clause in their policies leads to a demand for more information on the subject than has heretofore been available.

One of the methods of computing the extra premiums necessary for this purpose has been to form Commutation Columns representing the combined benefits by death and total disability.

The premiums computed from these columns for the combined benefits are then compared with the ordinary premiums, and the difference represents the extra premium required for the disability benefits.

Thus far, in this country, the bases taken have been the American Experience Table of Mortality for the benefit by death, and for those of disability, the Experience of the Knights of the Maccabees, a fraternal society of the United States. This experience had been tabulated, adjusted, and published in a book entitled *Tables Based on N. F. C. and Am. Exp.* by Landis and Eldridge, at pp. 175–184.

The matter was further developed by Franklin B. Mead, actuary, in a paper read before the American Life Convention in 1909, entitled "The Measure of Risk and Liability under the Total and Permanent Disability Benefits in Life Insurance." This paper was published in the *Report of the American Life Convention* of that year and also in pamphlet form. These books are referred to as containing some of the figures necessary to be used in the computations which follow.

For the purpose of combination with the American Table of Mortality, it is best to take the radix of the column of lives in existence at age 20, as shown by that table, namely, 92,637. A column designated as l_x^a (meaning active or able lives at age x) is then formed by multiplying the numbers in column l_x by the rate of mortality q_x of the American Table to form a column designated as d_x^a, and again by the rate of total disablement i_x

of the Maccabees Table to form a column designated as d_x^i. The numbers in these two columns at age 20 are added together and deducted from the number in column l_x at age 20, giving the number in column l_x^a at age 21, and so on for each subsequent age. It will, of course, be apparent that there is a decrement in the number of lives from both sources, so that the column l_x^a runs out at age 81, when all have either died or become totally and permanently disabled.

These columns, as computed, are shown in Table A annexed hereto.

From these figures the commutation columns are formed in the usual manner, namely: $D_x^a = l_x^a \cdot v^x$, N_x^a = the summation of D_x^a from the last age backward to age 20, $C_x^a = d_x^a \cdot v^{x+1}$, M_x^a = the summation of C_x^a.

$C_x^i = d_x^i \cdot v^{x+1}$, M_x^i = the summation of C_x^i.

The numbers at any age in the two columns M_x^a and M_x^i added together make the complete $M^a{}_x^i$, corresponding to D_x^a and N_x^a at the same age. They are separately computed so as to give in column M_x^i the probabilities of disablement by themselves. Hence when the benefit of total disability is payable by instalments, the values may be arrived at by multiplying the figures in column M_x^i by the instalment factor.

If it is desired to limit the total disability to happen before any certain age, say 65, it is necessary to have new commutation columns.

The initial column l_x^a will be the same as before up to age 64, but at age 65 the decrement by i_x ceases and only the decrement q_x for death is used. This gives new figures in columns l_x^a and d_x^a from age 66 downward, and, in fact, extends the column down to age 95 when all are dead, while column d_x^i ceases altogether at age 65.

From these new figures, the new commutation columns are computed as before, but as will readily be seen, column M_x^a will extend from age 20 to age 95, while column M_x^i will cease at age 64.

If it is desired to limit the occurrence of the total disability to any other age, as 55, 60, or 70, new commutation columns must be computed in a similar manner.

The preliminary columns for lives, deaths, and disabilities after age 60, to cease at age 65 and 60, are given in Table B annexed hereto.

TABLE A

LIVES, DEATHS, AND DISABLEMENTS TO AGE 80

AMERICAN AND MACCABEES EXPERIENCE

Age	l_x^a	d_x^a	d_x^i	Age	l_x^a	d_x^a	d_x^i
20	92 637	723	11	60	55 297	1476	459
21	91 903	722	12	61	53 362	1542	547
22	91 169	721	14	62	51 273	1604	662
23	90 434	720	16	63	49 007	1676	793
24	89 698	718	19	64	46 538	1715	1024
25	88 961	718	22	65	43 799	1752	1294
26	88 221	717	26	66	40 753	1781	1634
27	87 478	716	28	67	37 338	1779	2039
28	86 734	718	29	68	33 520	1743	2487
29	85 987	718	30	69	29 290	1663	2922
30	85 239	717	32	70	24 705	1532	3268
31	84 490	720	32	71	19 905	1347	3441
32	83 738	720	33	72	15 117	1115	3362
33	82 985	723	34	73	10 640	853	3000
34	82 228	727	36	74	6 787	591	2392
35	81 465	729	39	75	3 804	359	1655
36	80 697	733	42	76	1 790	182	932
37	79 922	738	45	77	676	75	419
38	79 139	745	48	78	182	22	134
39	78 346	751	51	79	26	0	22
40	77 544	759	54	80	4	0	4
41	76 731	768	56	81	0	0	0
42	75 907	778	58				
43	75 071	789	62				
44	74 220	804	64				
45	73 352	819	66				
46	72 467	838	68				
47	71 561	858	71				
48	70 632	884	74				
49	69 674	913	79				
50	68 682	947	85				
51	67 650	983	94				
52	66 573	1024	105				
53	65 444	1069	117				
54	64 258	1118	135				
55	63 005	1170	161				
56	61 674	1226	198				
57	60 250	1286	245				
58	58 719	1347	296				
59	57 076	1410	369				

TABLE B

LIVES, DEATHS, AND DISABLEMENTS AFTER AGE 60

DISABILITY CEASING AT 65				DISABILITY CEASING AT 60		
Age	l_x^a	d_x^a	d_x^i	l_x^a	d_x^a	Age
60	55 297	1476	459	55 297	1476	60
61	53 362	1542	547	53 821	1554	61
62	51 273	1604	662	52 267	1636	62
63	49 007	1676	793	50 631	1719	63
64	46 538	1715	1024	48 912	1804	64
65	43 799	1757		47 108	1890	65
66	42 042	1837		45 218	1976	66
67	40 205	1916		43 242	2060	67
68	38 289	1991		41 182	2142	68
69	36 298	2060		39 040	2216	69
70	34 238	2122		36 824	2283	70
71	32 116	2173		34 541	2337	71
72	29 943	2207		32 204	2374	72
73	27 736	2224		29 830	2392	73
74	25 512	2220		27 438	2388	74
75	23 292	2198		25 050	2364	75
76	21 094	2158		22 686	2321	76
77	18 936	2103		20 365	2262	77
78	16 833	2033		18 103	2187	78
79	14 800	1949		15 916	2097	79
80	12 851	1856		13 819	1996	80
81	10 995	1743		11 823	1875	81
82	9 252	1612		9 948	1734	82
83	7 640	1463		8 214	1573	83
84	6 177	1305		6 641	1404	84
85	4 872	1147		5 237	1234	85
86	3 725	989		4 003	1064	86
87	2 736	828		2 939	891	87
88	1 908	660		2 048	710	88
89	1 248	492		1 338	530	89
90	756	342		808	367	90
91	414	218		441	235	91
92	196	122		206	131	92
93	74	51		75	55	93
94	23	16		20	17	94
95	7	7		3	3	95
96	0	0		0	0	96

From the commutation columns formed as above indicated, the reserves for the combined benefits of death and total disability may be computed by the general formula, to wit: From the single premium at the advanced age deduct the net present value of all the future premiums. For this purpose it will be necessary to compute the single premiums and temporary annuities required. This may be done by the usual formulae employed in life insurance.

A comparison of the reserves thus found with the ordinary reserves gives the reserves for the extra premiums for total disability. Where, however, the clause in the policy only extends to the cessation of premiums, the reserves must be approximated by the proportion which the extra premium for cessation of payments bears to the extra premium for the complete benefit. Thus if the extra premium for immediate payment in case of total disability is 0.60 and that for cessation of payments is 0.20, the reserve for the latter would be twenty-sixtieths, or one-third of the reserve found for the former. These approximations will, I think, be close enough for practical purposes. I apprehend that by this method there will be no negative values for reserves.

ON THE CONSTRUCTION AND GRADUATION OF A RURAL LIFE TABLE

By H. L. RIETZ and C. H. FORSYTH

I. INTRODUCTION

In the July number of the *Journal of the Institute of Actuaries* for 1908, Mr. George King gives a method for constructing tables of mortality from census reports, which consists largely in an application of Sprague's osculatory method of interpolation. The purposes of the present paper are, (*a*) to apply osculatory interpolation to the construction of a table of mortality based on data obtained from the most stationary part of the population of the registration district of the United States for the year 1900; (*b*) to graduate the table thus obtained to Makeham's formula fitted by the method of moments; (*c*) to compare the resulting mortality rates for different ages with the rates given in the American Life Tables based on census returns from 1830 to 1860.

It is fairly well established* that rates of mortality for the decade beginning with 1890 show a considerable decrease in comparison with previous decades for ages less than 60 approximately and a corresponding increase for ages over 60. This appears to be true not only of the United States but of other countries where medical science has advanced materially and modern methods of sanitation are in use. The mortality table presented in the present paper seems to confirm this belief.

Meech's American Life Tables were constructed from the census reports covering the period from 1830 to 1860. It thus seems worth while to construct tables from recent census reports.† We hope to use the data of the census for 1910 for further tables as soon as available.

Special attention is called to the closeness of the fit by Makeham's formula. In fact, it seems that one of the most interesting

* *Bulletin 30*, Committe of One Hundred on National Health, Irving Fisher, p. 26; *Vital Statistics of Massachusetts*, 1856–95, p. 755; *Trans. Act. Soc. of A.*, 1902, addresses by President Oscar B. Ireland, pp. 233, 356; *Vital Statistics of Massachusetts*, for 1897, Samuel W. Abbott, M.D., with a life table based upon the experience of the five-year period, 1893–97.

† J. W. Glover, *Quarterly Publications of the American Statistical Association*, New Series No. 90, Vol XII, pp. 1–25, June, 1910.

results of the work consists in the fact that, as far as we can find, no table has been fitted so closely by Makeham's formula as this one.

Possible reasons for this closeness of fit are given among the conclusions, Sec. 5.

II. SOURCE OF DATA

The source of data used in the construction of this table is the United States Census Report for 1900. A single census report is used in order that the figures may furnish as recent information as possible. To get the most reliable figures, we have selected those collected from the registration area* which for 1900 consisted of the District of Columbia and the following nine states: Maine, Connecticut, Massachusetts, Michigan, New Hampshire, New Jersey, New York, Rhode Island, and Vermont. In such an area the registration is reliable to the extent that the deviations between all official registration records and those obtained from census enumeration generally fall within 10 per cent of the latter.

To lessen as much as possible the unavoidable influences of immigration, only the rural portion of the registration area is used, although, in the opinion of the census authorities, some degree of reliability of figures is thereby sacrificed. The figures for mortality are taken from *Vital Statistics*, Part 2, p. 60, but we are indebted to Mr. Durand, the director of the census, for the population data, which he sent us in its only available form, that of manuscript. The original data for both sexes follow:

Age x	l_x	d_x	Age x	l_x	d_x
Under 1	143369	16836	40–44	448606	3748
1–2	133240	3598	45–49	387655	3936
2–3	138513	1683	50–54	345949	4380
3–4	141133	1139	55–59	293916	5347
4–5	140549	736	60–64	252629	6445
5–9	692469	2520	65–69	200267	7981
10–14	657610	1794	70–74	145809	8932
15–19	637803	2881	75–79	94701	9253
20–24	620865	3727	80–84	48712	7338
25–29	574559	3931	85–89	18087	3986
30–34	522055	3552	90–94	4582	1526
35–39	489081	3782	95–	1031	422

* "The registration area is made up of states having laws the results of whose operation have been accepted as giving practically complete mortality returns." —*Mortality Statistics*, 1905, p. 5.

III. OSCULATORY INTERPOLATION

We assume that the table of mortality should end at age 105 and therefore let $l_{105}=q_{105}=0$.

To interpolate within quinquennial periods for each age both for l_x and d_x, the data are changed into columns of T_x, where each T_x represents the total number at and above age x. This change is suggested by the form in which the original data are given.

The method of interpolation is that of Dr. Sprague. The general problem is to find a curve $y_x=f(x)$ suitable for interpolation between any two ordinates of the set: $y_0, y_1, y_2, \ldots . y_n$. In this method of Dr. Sprague, if we wish to interpolate by fifth differences between two ordinates, say y_2 and y_3, a curve $y_x=f_1(x)$, where $f_0(x)$ is a polynomial of the fourth order, is passed through the five points y_0, y_1, y_2, y_3, y_4. Likewise, a curve $y_x=f_1(x)$ of the fourth order is passed through y_1, y_2, y_3, y_4, y_5. To secure smoothness at the intersections y_2 and y_3, these curves are required to have equal slopes and curvatures at y_2 and likewise equal slopes and curvatures at y_3. Then we impose the conditions that the interpolation curve $y_x=f(x)$ shall pass through y_2 and y_3, and have the same slopes and curvatures at y_2 and y_3 as the partial interpolation curves $y_x=f_0(x)$ and $y_x=f_1(x)$.

The general equation of the fourth order through the five points y_0 to y_4 is

$$y_x=y_0+x\Delta y_0+\tfrac{1}{2}(x^2-x)\Delta^2 y_0+\tfrac{1}{6}(x^3-3x^2+2x)\Delta^3 y_0+ {}_{2}^{1}{}_{4}(x^4-6x^3+11x^2-6x)\Delta^4 y_0.$$

The conditions that the slopes and curvatures mentioned above are equal, are equivalent to the conditions that the corresponding first derivatives are equal and the corresponding second derivatives are equal. The derivatives at y_2 and y_3 are:

$$\frac{dy_2}{dx}=\Delta y_0+\tfrac{3}{2}\Delta^2 y_0+\tfrac{1}{3}\Delta^3 y_0-{}_{1}^{1}{}_{2}\Delta^4 y_0$$

$$\frac{d^2y_2}{dx^2}=\Delta^2 y_0+\Delta^3 y_0-{}_{1}^{1}{}_{2}\Delta^4 y_0$$

$$\frac{dy_2}{dx}=\Delta y_0+\tfrac{5}{2}\Delta^2 y_0+{}_{6}^{1}{}^{1}\Delta^3 y_0+\tfrac{1}{4}\Delta^4 y_0-{}_{1}^{1}{}_{2}\Delta^5 y_0$$

$$\frac{d^2y_3}{dx^2}=\Delta^2 y_0+2\Delta^3 y_0+1\tfrac{1}{2}\Delta^4 y_0-{}_{1}^{1}{}_{2}\Delta^5 y_0.$$

We now assume the equation of the interpolation curve to be

$$y_{2+x}=y_2+ax+bx^2+cx^3+dx^4+ex^5$$

which, according to the conditions imposed, goes into the form

$$y_{2+x}=y_2+x\Delta y_0+\tfrac{1}{2}(3x+x^2)\Delta^2 y_0+\tfrac{1}{6}(2x+3x^2+x^3)\Delta^3 y_0-\tfrac{1}{24}(2x+x^2-2x^3-x^4)\Delta^4 y_0+\tfrac{1}{24}(7x^3-12x^4+5x^5)\Delta^5 y_0.$$

If we give to x the values $\frac{0}{5}$, $\frac{1}{5}$ . . . . $\frac{5}{5}$ and difference the resulting six equations five times, we form the leading differences by means of which to interpolate four values of y between y_2 and y_3, thus dividing the interval into five minor intervals. The resulting leading differences are as follows:

$$\delta y=\frac{\Delta y_0}{5}+8\frac{\Delta^2 y_0}{5^2}+11\frac{\Delta^3 y_0}{5^3}-11\frac{\Delta^4 y_0}{5^4}+\frac{\Delta^5 y_0}{5^4}$$

$$\delta^2 y=\quad 1\frac{\Delta^2 y_0}{5^2}+\ 6\frac{\Delta^3 y_0}{5^3}+\ 1\frac{\Delta^4 y_0}{5^4}+3\frac{\Delta^5 y_0}{5^4}$$

$$\delta^3 y=\quad\quad 1\frac{\Delta^3 y_0}{5^3}+\ 4\frac{\Delta^4 y_0}{5^4}-3\frac{\Delta^5 y_0}{5^4}$$

$$\delta^4 y=\quad\quad\quad 1\frac{\Delta^4 y_0}{5^4}-2\frac{\Delta^5 y_0}{5^4}$$

$$\delta^5 y=\quad\quad\quad\quad 5\frac{\Delta^5 y_0}{5^4}$$

To apply this method of interpolation, the observed values of T_x are differenced, the differences divided by the appropriate power of five as shown above and then the leading differences themselves formed. Each quinquennial interval will have its own set of differences derived from the differences of an age ten years younger. The interpolated values of y for each interval are formed by the continued addition of the differences in the usual way and there is a good check on the computation because the next higher quinquennial value of y given in the original data must be reproduced at each stage. The work is performed to two more decimal places than are to be finally retained. Columns of l_x and d_x are formed from the complete tables of T_x and the corresponding column of q_x are calculated. Values of q_x for ages 5 to 9, 95 to 99, 101 to 104 are determined approximately by graphical interpolation. By means of the values of q_x, the mortality table itself is constructed, taking $l_{10}=100{,}000$ as is customary.

The results of osculatory interpolation yield the following values from age 19 to 85:

Age	l_x	d_x	q_x	Age	l_x	d_x	q_x
19	96932	506	.0052150	53	73857	986	.0133458
20	96426	530	.0054956	54	72871	1053	.0144469
21	95896	555	.0057858	55	71818	1123	.0156436
22	95341	575	.0060356	56	70695	1202	.0170083
23	94766	594	.0062667	57	69493	1276	.0183661
24	94172	608	.0064542	58	68217	1336	.0195864
25	93564	621	.0066386	59	66881	1389	.0207649
26	92943	645	.0068309	60	65492	1444	.0220520
27	92298	641	.0069405	61	64048	1500	.0234251
28	91657	636	.0069433	62	63548	1572	.0251328
29	91021	625	.0068661	63	60976	1672	.0274256
30	90396	614	.0067894	64	59304	1795	.0302705
31	89782	600	.0066795	65	57509	1914	.0332838
32	89182	595	.0066728	66	55595	2036	.0366137
33	88587	604	.0068183	67	53559	2149	.0401218
34	87983	622	.0070683	68	51410	2246	.0436839
35	87361	641	.0073360	69	49164	2334	.0474736
36	86720	660	.0076117	70	46830	2415	.0515753
37	86060	674	.0078292	71	44415	2489	.0560422
38	85386	678	.0079423	72	41926	2566	.0612140
39	84708	675	.0079661	73	39360	2649	.0672938
40	84033	675	.0080311	74	36711	2728	.0743162
41	83358	676	.0081138	75	33983	2785	.0819508
42	82682	683	.0082574	76	31198	2823	.0904921
43	81999	700	.0085313	77	28375	2821	.0994038
44	81299	723	.0088928	78	25554	2769	.1083712
45	80576	749	.0093007	79	22785	2682	.1176903
46	79827	779	.0097629	80	20103	2587	.1286742
47	79048	809	.0102297	81	17516	2485	.1418913
48	78239	830	.0106125	82	15031	2338	.1555672
49	77409	850	.0109752	83	12693	2339	.1842822
50	76559	875	.0114050	84	10354	1860	.1795806
51	75686	896	.0118411	85	8994	1639	.1929476
52	74790	933	.0124718				

IV. GRADUATION TO MAKEHAM'S FORMULA

The graduation of a table of mortality to Makeham's formula consists in representing the number living at age x by the well-known formula

$$l_x = ks^x g^{c^x}, \qquad (1)$$

where the parameters k, s, g, and c are to be determined from the observed data. For a number of tables, these parameters have been determined. It has been found, in these cases, that formula (1) represents the mortality with sufficient accuracy for many practical purposes. The present table is to be graduated from age 19 to 85.

The methods that may be employed to obtain the parameters differ materially. If, in any case, the formula fits perfectly the table to be graduated, then the matter is very simple; for, only four corresponding values of x and l_x need be substituted in (1) to obtain four equations in k, s, g, and c. The solution of these four equations gives the desired values. But as a perfect fit to the statistical data for ages from, say 19 to 85, is not to be expected with a function involving only four parameters, the problem of determining the best or even a satisfactory set of values for k, s, g, and c is one of real difficulty.

One method of dealing with the problem consists of repeating the determinations of the parameters from different sets of four values of x and l_x, and then averaging the results obtained. This method has given some good results, but it is fundamentally a process of trial and repetition. It offers little in the way of a systematic method of approaching the problem of graduation of data for future tables. This method is well illustrated in the Text-book and in Hunter's graduation of the American Experience Table.

The method of least squares may next be mentioned. The disadvantage of this method lies in the fact that it is very laborious.*

A third method, the one that we employ in this paper, is called the method of moments. This method is systematic and not nearly so laborious as the method of least squares, and it is based on principles that are satisfactory for many of the purposes of graduation.

The principles involved in graduation by the method of moments are developed in a series of memoirs by Karl Pearson.† A brief account of these principles is perhaps most available to members of this society in Hardy's *Construction of Tables of Mortality*, and in P. Elderton's *Frequency Curves and Correlation*. Perhaps it should be said that the method has been applied to the American Experience Table.‡

* The application of this method is illustrated by the graduation of a French mortality table—*Tables de mortalité du comité des compagnies d'assurances à primes fixes sur la vie*, Paris, 1895.

† Pearson, *Philosophical Transactions of the Royal Society*, Vol. 186A, Part 1, p. 343, 1895; Vol. 197A, p. 443, 1901; *Biometrika*, Vol. I, Part III, p. 298.

‡ J. W. Glover, *Transactions of the Actuarial Society of America*, Vol. VII, p. 339, 1901.

We shall next proceed to a statement of the working principle of the method of moments as applied to our problem:

The process of making the graduation is simplified by taking the logarithms of each member of (1). This gives

$$L_x = K + xS + Gc^x, \tag{2}$$

where the capitals represent the logarithms of the corresponding lower case letters.

Take logarithms of the number living at each age in the table to be graduated. Consider as the curve to be fitted the best curve that can be obtained by representing these logarithms with reference to coördinate axes with the ages as abscissas and the corresponding logarithms as ordinates.

We may next consider a theoretical curve L_x given by (2) referred to the same axes.

We define the ith moment, of an area, about any line as an axis, by the formula $\underset{\Delta A = 0}{\text{Limit}} \sum x^i \Delta A$, where ΔA is an element of area whose perpendicular distance from the axis is x, and where the summation, indicated by Σ, is to extend over the entire area concerned. The working principle of the method of moments is to equate the zeroth, first, second, third, etc., moments of the area under the curve to be fitted to the corresponding moments of the area under the theoretical curve, until we obtain as many equations as there are parameters to be determined.

For the curve to be fitted, close approximations to the numerical values of the moments are obtained from the data by means of some quadrature formula and the degree of accuracy depends upon the quadrature formula selected. For our problem, it is necessary to compute only the zeroth, first, second, and third moments, as these will give four equations when the numerical values are equated to the moments of the area under the curve given by (2).

To outline the details of the application, we may write (2), after slight transformations that introduce the range $l = 19\text{–}85$ and n defined by $c = e^{\frac{2n}{l}}$, and place the origin at the middle of the range, in the form

$$L_x = K + x\frac{S}{l} + Gc^x \tag{3}$$

The moments derived from the theoretical curve, we obtain by the integrations

$$\int_{-\frac{l}{2}}^{\frac{l}{2}} L_x dx\,,\quad \int_{-\frac{l}{2}}^{\frac{l}{2}} xL_x dx\,,\quad \int_{-\frac{l}{2}}^{\frac{l}{2}} x^2 L_x dx\,,\quad \int_{-\frac{l}{2}}^{\frac{l}{2}} x^3 L_x dx\,.$$

These integrations are easily performed and the results expressed in hyperbolic functions. We may then write the results of equating moments in the following form:

$$a_0 = K + G\frac{\sinh n}{n}\,,$$

$$a_1 = S + 6G\left\{\frac{\cosh n}{n} - \frac{\sinh n}{n^2}\right\},$$

$$a_2 = K + 3G\left\{\frac{\sinh n}{n} - \frac{2\cosh n}{n^2} + \frac{2\sinh n}{n^3}\right\},$$

$$a_3 = S + 10G\left\{\frac{\cosh n}{n} - \frac{3\sinh n}{n^2} + \frac{6\cosh n}{n^3} - \frac{6\sinh n}{n^4}\right\},$$

where

$$a_0 = \frac{A}{l} = \frac{1}{l}\int_{-\frac{l}{2}}^{\frac{l}{2}} L_x d_x = Kl + \frac{Gl}{2n}(\epsilon^n - \epsilon^{-n}),$$

$$a_1 = \frac{12A\mu_1}{l^2}\,,$$

$$a_2 = \frac{12A\mu_2}{l^3}\,,$$

$$a_3 = \frac{80A\mu_3}{l^4}\,,$$

in which A and $A\mu_i$ represent the area and ith moment respectively:

$$\text{Let } \beta = \frac{a_3 - a_1}{a_2 - a_0}\,;\ \text{then } \beta = 4\frac{\left(\frac{20\mu_3}{l^3} - \frac{3\mu_1}{l}\right)}{\frac{12\mu_2}{l^2} - 1}\,;\ \text{and}$$

$$\tanh n = \frac{2n^3 + 3\beta n^2 + 30n}{\beta n^3 + 12n^2 + 3\beta n + 30}\,.$$

Express tanh n in exponentials, and solve for ϵ^{2n}. This gives

$$\epsilon^{2n}-\frac{(\beta+2)n^3+3(\beta+4)n^2+3(\beta+10)n+30}{(\beta-2)n^3-3(\beta-4)n^2+3(\beta-10)n+30}=\epsilon^{2n}-\frac{N}{D}=0.$$

The value of "n" of this last equation is approximated by Newton's method where "h," the needed correction to a trial value "a," is determined from

$$h=-\frac{f(a)}{f^1(a)}=\frac{D\epsilon^{2n}-N}{\frac{dN}{dn}-\frac{Nd}{Dd}\frac{D}{n}-2\epsilon^{2n}D},$$

where

$$N=(\beta+2)n^3+3(\beta+4)n^2+3(\beta+10)n+30,$$
$$D=(\beta-2)n^3-3(\beta-4)n^2+3(\beta-10)n+30.$$

For close approximations of the area and moments, we use Weddle's quadrature formula for p elements:

$$\int_{x_o}^{x_{6p}} z\,dx=\tfrac{3}{10}[z_0+z_2+\ \ldots\ z_{6p}+2(z_6+z_{12}+\ \ldots\ z_{6p-6})+5(z_1+z_5+z_7\ \ldots\ z_{6p-1})+6(z_3+z_9+\ \ldots\ z_{6p-3})],$$

where the ordinates are $z=L_x$ for the area and x^iz for the moments. The final constants are found to be:

$$n=3.489009393$$
$$G=-.027099489$$
$$K=4.900693 13$$
$$\frac{S}{l}=-.002649892 84$$
$$c=1.11151901$$
$$\frac{1}{c}=.899669722$$

The function thus takes the form:

$$L_x=4.900693 13-.002649892 84x-(.027099489 0)(1.11151901)^x,$$

which gives the logarithm of the number living at age x.

V. CONCLUSIONS

For the purpose of discussing the graduation by Makeham's formula, the criterion for a successful fit used by Professor Pearson in *Phil. Mag.*, 1900, Vol. CCXVI, is applied to the graduations of the American Experience table by Arthur Hunter and by J. W. Glover. The criterion may be represented by $X=\sqrt{\Sigma\frac{(y-y_0)^2}{y}}$ where $(y-y_0)$ is the deviation of y, the theoretical value, and y_0, the observed value. This test is applied not only to the entire table but to the lower and upper halves separately for the purpose of comparison. The values of X and the average deviations (A. D.) of the whole table are

	Whole Table	Lower Half	Upper Half	A. D.
Hunter	9.468	.966	9.419	114
Glover	6.606	2.057	6.277	132
Present table	2.264	1.105	1.976	54

The criterion not only indicates a remarkably close fit but that the fit is unusually smooth throughout the table. The mean of the absolute deviations of the logarithmic columns is .00047 compared with .00116 obtained by Professor Pearson in a similar graduation in the article cited in section IV. Comparison of rates of mortality of this table and of Meech's tables shows a decrease for ages below seventy approximately, and an increase for higher ages.

The use of rural data may probably be properly assigned as the reason why this decrease reaches the age 70 instead of 60. If Dr. Sprague's method of interpolation does not account for much of the closeness of fit, we have here a strong suggestion that Makeham's formula is something more than an empirical formula, because it is applied in the present case to the unselected data of the general population.*

The graduated table, deviations, and resulting rates of mortality follow:

* It should perhaps be stated that, in the present paper, we leave out of consideration the method of collection of data, and concern ourselves solely with their mathematical treatment.

Age	l_x'	d_x	$l_x'-l_x$	q_x	Age	l_x'	d_x	$l_x'-l_x$	q_x
19.	97121	612	189	.0063014	53.	73777	1014	80	.0137441
20.	96509	609	73	.0063103	54.	72763	1061	108	.0145816
21.	95900	609	4	.0063504	55.	71702	1114	116	.0155365
22.	95291	607	50	.0063700	56.	70588	1171	107	.0165892
23.	94684	606	82	.0064002	57.	69417	1232	76	.0177478
24.	94078	606	94	.0064415	58.	68185	1298	32	.0190364
25.	93472	606	92	.0064832	59.	66887	1360	6	.0203328
26.	92866	606	77	.0065255	60.	65517	1446	25	.0220706
27.	92260	607	38	.0065792	61.	64071	1527	23	.0238329
28.	91653	607	4	.0066228	62.	62544	1613	4	.0257898
29.	91046	609	25	.0066889	63.	60931	1704	45	.0279661
30.	90437	612	41	.0067671	64.	59227	1799	77	.0303747
31.	89825	614	43	.0068355	65.	57428	1898	81	.0330501
32.	89211	617	29	.0069162	66.	55530	2000	65	.0360166
33.	88594	621	7	.0070095	67.	53530	2103	29	.0392860
34.	87973	625	10	.0071045	68.	51427	2208	17	.0429346
35.	87348	632	13	.0072354	69.	49219	2311	55	.0469534
36.	86716	638	4	.0073573	70.	46908	2411	78	.0513985
37.	86078	645	18	.0074932	71.	44497	2506	82	.0563184
38.	85433	654	47	.0076551	72.	41991	2585	65	.0615608
39.	84779	664	71	.0078321	73.	39406	2680	46	.0680099
40.	84115	675	82	.0080247	74.	36726	2733	15	.0744159
41.	83440	688	82	.0082454	75.	33993	2779	10	.0817521
42.	82752	702	70	.0084832	76.	31214	2803	16	.0897994
43.	82050	718	51	.0087508	77.	28411	2805	36	.0987294
44.	81332	736	33	.0090493	78.	25606	2778	52	.1084902
45.	80596	756	20	.0093801	79.	22828	2721	43	.1191957
46.	79840	778	13	.0097445	80.	20107	2626	4	.1306013
47.	79062	802	14	.0101439	81.	17481	2524	35	.1443853
48.	78260	830	21	.0106057	82.	14957	2365	74	.1581199
49.	77430	860	21	.0111068	83.	12592	2184	101	.1734435
50.	76570	893	11	.0116625	84.	10408	1982	54	.1904304
51.	75677	930	9	.0122891	85.	8426	....	68	.....
52.	74747	970	43	.0129771					

Notice that the series of deviations of this table winds around through zero 11 times; that of Glover does so 4 times, and Hunter's 3 times.

The present paper was practically completed at the time of the publication of the recent paper by Glover, to which reference has been made.* One point by way of comparison may be of interest; namely, the mortality rates of our table are less than those obtained by Glover except for ages at the ends of the table.

It may be worth while to recall here that Glover's paper deals with the general male population of the registration district, while our paper treats only the rural element—both male and female—of the population of that district.

* *Loc. cit.*, pp. 1–25.

ON THE NATURE OF MATHEMATICAL LAWS OF MORTALITY WITH SPECIAL REFERENCE TO MAKEHAM'S LAW TOGETHER WITH NEW TABLES FOR INSURANCES ON ONE TO FIVE LIVES

By R. MONTAGUE WEBB

INTRODUCTORY

My object is threefold: in the first place, as one of our aims is the advancement of actuarial mathematics, I wish to open up to our students an avenue that has been but little trodden, which nevertheless leads to very much that is at the foundation of our science. Secondly, I have had occasion to compute annuity values on four joint lives with the object of calculating insurance premiums and reserves therefrom; the computing of the values involving from two to five lives therefore seemed to me of value now that joint life insurances and reversionary annuities are becoming popular. As a third object, I trust it will not be considered superfluous if I briefly outline the method of these computations, adding such auxiliary tables as will aid in the calculation of similar annuity values, either for a greater number of lives or with some other involved rate of interest.

With the evident utility of the adoption of an assumed mathematical law of mortality and the close agreement of observed facts to the formula of Gompertz as modified by Makeham, it becomes at least interesting and instructive to trace the lines of thought that did lead and may lead to these results. I am sure, moreover, that such a study is not merely interesting and instructive, but a necessity to such of us as may select as a special subject the contemplation of mortality—or vitality—which lies at the root of our practice and forms so great a part of the foundation of our theory. I have met, and we all know, men in the profession to whom the formulas of Gompertz and Makeham are, in their nature and essence, absolutely unknown—considering them as the clever guesses of clever men: academical and of little practical importance. Others there are who take the deduced results of these laws as matters of course, using tables of equal-age annuity values with the same confidence as they use the American single

life table, accepting the results but knowing nothing of the operations. For some reason, or possibly by chance only, the literature of our societies is practically devoid of reference to the initial steps whereby a student may gain an entrance to this highly interesting field of thought. Much has been deduced from the acceptance of Makeham's theory; and much harm has been the result of thus misnaming Makeham's improved approximation. It is, therefore, my intention to outline a course that might very well lead to an intelligent understanding of the various formulas that have been obtained, as well as to indicate the process that will enable the student to undertake independent investigation.

The logical method by which to present this subject, I know, would be to set forth the evolution of the resulting formulas from the primary ideas. Nevertheless, I propose to adopt another way; with your permission, I will first demonstrate one of the uses to which Makeham's formula can be put in the matter of facilitating joint life calculations.

PART I

1. The expression for the number living at age x is, according to Makeham's law, $ks^x g^{c^x}$ in which k, s, c, and g are constant. This form appears, perhaps, somewhat forbidding, but I hope to show later that it is the mathematical expression of a comparatively simple idea and the result of a simple sequence of investigation. Put in logarithmic form, the expression becomes

$$\log l_x = \log k + x \log s + c^x \log g$$

which to some minds will seem not so complicated. The constants that occur may be found from values of the number living, and in the case of the American experience table, their logarithms are:

$\log c$	0.04579609
$\log s$ . . .	−0.00329686
$\log g$ . . .	−0.00013205
$\log k$	5.03370116

all to base 10.

The Napierian logarithm of s is $-.0075913053$.

2. The advantage of the formula in the calculation of joint life annuities by the method I have here adopted rests on the

following property. I will first call attention to the simple relations.

$$ {}_tp_x = \frac{l_{x+t}}{l_x} \qquad {}_tp_{xyz} = {}_tp_x \cdot {}_tp_y \cdot {}_tp_z \text{ and } \Sigma v^t {}_tp_x = a_x . $$

Using three lives for example, as the same idea applies to any number of joint lives:

since $l_x = ks^x g^{c^x}$

then $l_{x+t} = ks^{x+t} g^{c^{x+t}}$

and ${}_tp_x = s^{\,} g^{c^x(c^t-1)}$

hence ${}_tp_{xyz} = s^{3t} g^{(c^x+c^y+c^z)(c^t-1)}$

and if $c^x + c^y + c^z = c^u$

$$ {}_tp_{xyz} = s^{3t} g^{c^u(c^t-1)} $$

and $$ v^t {}_tp_{xyz} = v^t s^{3t} g^{c^u(c^t-1)} $$
$$ = (vs^2)^t s^t g^{c^u(c^t-1)} $$

Writing $\grave{v}$ for vs^2

$$ v^t {}_tp_{xyz} = \grave{v}^t s^t g^{c^u(c^t-1)} $$

Hence, provided $c^u = c^x + c^y + c^z$ and $\grave{v} = vs^2$

$$ v^t {}_tp_{xyz} = \grave{v}^t {}_tp_u $$

and $a_{xyz} = \grave{a}_u$.

The advantages of this method are now apparent; for the value of $\grave{a}_u$ for every integral value of u may be calculated in the same manner as single life annuities no matter how many lives there may be involved: the number of lives affecting only the discount rate v which, in the case of three lives, is vs^2 where v is the required rate, say at $3\frac{1}{2}$ per cent interest—and in the case of m lives

$$ v = vs^{m-1} $$

3. The final tables I have given are on the basis of $3\frac{1}{2}$ per cent and v therefore is .96618357 and its logarithm $\bar{1}.9856596502$. Calculating $\grave{v}$ for values 2, 3, 4, and 5 of m from the relation:

$$ \log \grave{v} = \log v - \overline{m-1} \log s $$

The following are the discount and interest rates:

No. of Lives	Discount	Interest
1	.96618357	.035000
2	.9588766	.042887
3	.9516250	.050834
4	.94442796	.058842
5	.9372861	.066910

from which it is seen that there is an increase of approximately .008 per cent in the interest rate for each additional life.

4. The calculation of the annuity values for integral values of u may now be conducted by one of two methods:

(*a*) For machine work the formula

$$a_u = v\,p_u\,(1+a_{u+1})$$

is ready for use: the values of p_u will be found in my auxiliary Table I and are, of course, the same for any number of lives and any rate of interest. I have tabulated these in reverse order as the calculations proceed from the oldest age backward. The column $v\,p_u$ may be quickly prepared on the machine by setting up the value of v and multiplying by the successive values of p: the results being checked by multiplying the total of the p_u column which is 80.12773 by v which should also be the sum of the $v\,p_u$ column. The working sheet for the annuity values is then of the form:

THREE LIVES

u	$v\,p_u$	a_u
98	.23791	.23791
97	.34605	.42838
96	.38770	.55378
.	.	.
.	.	.
.	.	.

(*b*) By using the logarithmic method the formula becomes:

$$\text{og } a_u = \log\,(vp_u) + \log\,(1+a_{u+1})$$

The values log p_u and its differences I have also given in my auxiliary Table I, by means of which

$$\log (v' p_u) = \log v' + \log p_u$$

may be formed and checked.

The working sheet for the annuities will then be:

TWO LIVES

u	$\log v' p_u$	$\log (1+a_u)$	$\log a_u$	a_u
98	.3797028	.09333	.37970	.23972
97	.5424301	.15603	.63579	.43227
96	.5917917	.	.	.

The final age being 99, the initial value is $a_{98} = v' p_{98}$.

My Table III gives the values of a_u calculated for the values of v' corresponding to vs^{m-1} for $m = 1, 2, 3, 4$, and 5, that is for use in the case of any number of joint lives from 1 to 5.

5. It now only remains to determine the value of u in a_u. Referring to the original demonstration, the requisite relation for the three lives is that c^u shall equal $c^x + c^y + c^z$ and this relation holds for any number of lives. I have given, in my Table II, the values of the powers of c from 10 to 75, this range being sufficient. The differences are also given for exact calculations. It will be found more speedy to extract the values c^x, c^y, c^z and add them, although the relation $c^x + c^y + c^z = c^x(1 + c^{y-x} + c^{z-x})$ may be used: thus if the ages are 35, 40, and 47,

$$c^{37} + c^{40} + c^{47} = c^{37}(1 + c^3 + c^{10})$$

The practical use of the tables is then

First: Determine u from $c^u = c^x + c^y + \ldots\ldots$ Table II.

Second: Determine a_u from the correct column in Table III.

Third: Calculate the single or annual premium from the relations

$$A_x = 1 - d(1+a) \text{ and } P = (1+a)^{-1} - d$$

6. However, I am sure that a sufficient degree of accuracy will be attained by using the tables in the following manner:

Calculate u to the nearest integer and one-half: then take the corresponding annuity value. Reference to the one-life table will

then give the equivalent single age at which the office premium for the required plan may be quoted.

Taking for example a twenty payment life policy on three lives aged 30, 37, and 42:

$$c^{30} = 23.65$$
$$c^{37} = 49.48$$
$$c^{42} = 83.84$$
$$156.97 = c^{48}$$
$$a_{48} = 12.049$$
$$A = .55873$$
$$P = .04282$$

Otherwise:

$$a_{48} = 12.049 = a_{54\frac{1}{2}}$$

for a single life giving

$$A = .56022$$
$$P = .04309$$

The values of the annuities here determined are for use in the calculation of insurances and not primarily for annuity contracts. I can, however, see no reason for not using them in the premiums for continuous instalment contracts and such contracts as combine insurances and annuities.

Finally, the method I have outlined is to be commended in that it evades the calculation of commutation columns: it is thereby speedier. The auxiliary Table I is available for the calculation of similar annuity tables at any other rate, say 3 per cent, the value of v being changed. Moreover, it brings before our members a very practical application of a branch of our science that should receive greater attention from our students of insurance mathematics.

Table I. Auxiliary values of p_u, log p_u, and $-\Delta$ log p_u.

Table II. c^n and its differences.

Table III. a_u for one to five lives.

TABLE I

AUXILIARY TABLE FOR FORMATION OF JOINT LIFE ANNUITY VALUES

At any rate of interest on the basis of the American Table of Mortality: Makehamized

Age	p	log. p	Diff.	Age	p	log. p	Diff.
98...	.25000	.3979400		53..	.98351	.9927767	.0004390
97...	.36364	.5606673	.1627273	52..	.98438	.9931644	.0003877
96...	.40741	.6100289	.0493616	51..	.98520	.9935230	.0003586
95...	.46552	.6679358	.0579069	50..	.98592	.9938426	.0003196
94...	.50435	.7027302	.0347944	49..	.98658	.9941314	.0002888
93...	.53738	.7302840	.0275538	48..	.98714	.9943797	.0002483
92...	.57219	.7575422	.0272582	47..	.98769	.9946193	.0002396
91...	.60420	.7811810	.0236388	46..	.98816	.9948273	.0002080
90...	.63422	.8022408	.0210598	45..	.98858	.9950111	.0001838
89...	.66348	.8218371	.0195963	44..	.98897	.9951836	.0001725
88...	.69061	.8392331	.0173960	43..	.98933	.9953396	.0001560
87...	.71621	.8550386	.0158055	42..	.98962	.9954684	.0001288
86...	.73999	.8692230	.0141844	41..	.98992	.9955994	.0001310
85...	.76204	.8819779	.0127549	40..	.99016	.9957047	.0001053
84...	.78261	.8935446	.0115667	39..	.99039	.9958073	.0001026
83...	.80131	.9037995	.0102549	38..	.99058	.9958910	.0000837
82...	.61865	.9130087	.0092992	37..	.99078	.9959782	.0000872
81...	.83459	.9214745	.0083758	36..	.99095	.9960525	.0000743
80...	.84919	.9290046	.0075301	35..	.99109	.9961146	.0000621
79...	.86254	.9357797	.0067751	34..	.99122	.9961701	.0000555
78...	.87476	.9418885	.0061088	33..	.99136	.9962297	.0000596
77...	.88583	.9473505	.0054620	32..	.99145	.9962724	.0000427
76...	.89601	.9523124	.0049619	31..	.99156	.9963195	.0000471
75...	.90519	.9567419	.0044295	30..	.99164	.9963556	.0000361
74...	.91356	.9607373	.0039954	29..	.99171	.9963859	.0000303
73...	.92119	.9643497	.0036124	28..	.99180	.9964257	.0000398
72...	.92809	.9675879	.0032382	27..	.99186	.9964499	.0000242
71...	.93431	.9704911	.0029032	26..	.99191	.9964738	.0000239
70...	.93997	.9731151	.0026240	25..	.99197	.9964973	.0000235
69...	.94511	.9754823	.0023672	24..	.99202	.9965205	.0000232
68...	.94973	.9776020	.0021197	23..	.99206	.9965385	.0000180
67...	.95394	.9795203	.0019183	22..	.99209	.9965517	.0000132
66...	.95722	.9812369	.0017166	21..	.99213	.9965693	.0000176
65...	.96112	.9827773	.0015404	20..	.99216	.9965822	.0000129
64...	.96423	.9841787	.0014014	19..	.99219	.9965948	.0000126
63...	.96701	.9854303	.0012516	18..	.99221	.9966030	.0000082
62...	.96953	.9865616	.0011313	17..	.99224	.9966155	.0000125
61...	.97178	.9875663	.0010047	16..	.99226	.9966235	.0000080
60...	.97385	.9884904	.0009241	15..	.99227	.9966316	.0000081
59...	.97518	.9893077	.0008173	14..	.99229	.9966395	.0000079
58...	.97735	.9900514	.0007437	13..	.99231	.9966476	.0000081
57...	.97885	.9907157	.0006643	12..	.99232	.9966513	.0000037
56...	.98020	.9913155	.0005998	11..	.99233	.9966550	.0000037
55...	.98142	.9918565	.0005410	10..	.99235	.9966632	.0000082
54...	.98251	.9923377	.0004812				

TABLE II

CONSTANT TO DETERMINE THE EQUIVALENT SINGLE AGE

American Table of Mortality: Makehamized

$c = 1.1112$ $\log c = 0.04579609$

u]	c	Diff.	u	c	Diff.
10.....	2.87	.32	43.....	93.16	10.36
11.....	3.19	.35	44.....	103.52	11.51
12.....	3.54	.40	45.....	115.03	12.80
13.....	3.94	.44	46.....	127.83	14.21
14.....	4.38	.48	47.....	142.04	15.80
15.....	4.86	.54	48.....	157.84	17.55
16.....	5.40	.61	49.....	175.39	19.51
17.....	6.01	.66	50.....	194.90	21.67
18.....	6.67	.75	51.....	216.57	24.09
19.....	7.42	.82	52.....	240.66	26.76
20.....	8.24	.92	53.....	267.42	29.74
21.....	9.16	1.01	54.....	297.16	33.05
22.....	10.17	1.14	55.....	330.21	36.72
23.....	11.31	1.25	56.....	366.93	40.80
24.....	12.56	1.40	57.....	407.73	45.35
25.....	13.96	1.55	58.....	453.08	50.38
26.....	15.51	1.73	59.....	503.46	56.00
27.....	17.24	1.92	60.....	559.46	62.21
28.....	19.16	2.13	61.....	621.67	69.14
29.....	21.29	2.36	62.....	690.81	76.82
30.....	23.65	2.63	63.....	767.63	85.37
31.....	26.28	2.93	64.....	853.00	94.86
32.....	29.21	3.24	65.....	947.86	105.42
33.....	32.45	3.61	66.....	1,053.28	117.13
34.....	36.06	4.01	67.....	1,170.41	130.16
35.....	40.07	4.46	68.....	1,300.57	144.64
36.....	44.53	4.95	69.....	1,445.21	160.72
37.....	49.48	5.51	70.....	1,605.93	178.59
38.....	54.99	6.11	71.....	1,784.52	198.46
39.....	61.10	6.80	72.....	1,982.98	220.52
40.....	67.90	7.55	73.....	2,203.50	245.06
41.....	75.45	8.39	74.....	2,448.56	272.30
42.....	83.84	9.32	75.....	2,720.86	302.59

TABLE III

ANNUITY VALUES ON ONE TO FIVE LIVES AT EQUIVALENT SINGLE AGE WITH CHANGED RATE OF INTEREST

American Table: Makehamized: $3\frac{1}{2}$ per cent

Age	1 Life	2 Lives	3 Lives	4 Lives	5 Lives	Age
10	21.20911	18.47264	16.28505	14.51006	13.14341	10
11	21.12070	18.41342	16.24491	14.48231	13.03103	11
12	21.02895	18.35156	16.20257	14.45300	13.01050	12
13	20.93349	18.28676	16.15793	14.42188	12.98859	13
14	20.83415	18.21883	16.11083	14.38884	12.96502	14
15	20.73099	18.14783	16.06132	14.35389	12.93998	15
16	20.62384	18.07360	16.00924	14.31689	12.91336	16
17	20.51230	17.99577	15.95427	14.27758	12.88504	17
18	20.39639	17.91435	15.89642	14.23594	12.85473	18
19	20.27594	17.82934	15.83568	14.19195	12.82244	19
20	20.15074	17.74037	15.77171	14.14532	12.78801	20
21	20.02079	17.64742	15.70449	14.09601	12.75144	21
22	19.88589	17.55027	15.63382	14.04385	12.71255	22
23	19.74602	17.44891	15.55950	13.98878	12.67132	23
24	19.60075	17.34290	15.48130	13.93046	12.62742	24
25	19.45004	17.23219	15.39916	13.86882	12.58065	25
26	19.29386	17.11672	15.31301	13.80376	12.53107	26
27	19.13195	16.99643	15.22279	13.73522	12.47848	27
28	18.96406	16.87085	15.12789	13.66279	12.42262	28
29	18.79009	16.73987	15.02836	13.58634	12.36341	29
30	18.61040	16.60376	14.92426	13.50604	12.30085	30
31	18.42408	16.46185	14.81513	13.42132	12.23455	31
32	18.23121	16.31400	14.70081	13.33202	12.16421	32
33	18.03208	16.16040	14.58131	13.23824	12.08993	33
34	17.82577	16.00037	14.45613	13.13937	12.01141	34
35	17.61311	15.83440	14.32555	13.03575	11.92855	35
36	17.39354	15.66197	14.18904	12.92689	11.84104	36
37	17.16672	15.48284	14.04655	12.81254	11.74861	37
38	16.93281	15.29713	13.89796	12.69270	11.65129	38
39	16.69216	15.10491	13.74335	12.56738	11.54904	39
40	16.44399	14.90558	13.58211	12.43590	11.44133	40
41	16.18861	14.69933	13.41440	12.29860	11.32823	41
42	15.92590	14.48586	13.23989	12.15488	11.20925	42
43	15.65627	14.26560	13.05881	12.00509	11.08466	43
44	15.37909	14.03788	12.87066	11.84859	10.95395	44
45	15.09482	13.80323	12.67576	11.68570	10.81720	45
46	14.80362	13.56152	12.47395	11.51625	10.67433	46
47	14.50539	13.31261	12.26508	11.33999	10.52499	47
48	14.20019	13.05660	12.04921	11.15691	10.36927	48
49	13.88864	12.79398	11.82663	10.96730	10.20724	49
50	13.56993	12.52418	11.59661	10.77060	10.03844	50
51	13.24545	12.24785	11.36023	10.56723	9.86305	51
52	12.91504	11.96502	11.11706	10.35717	9.68112	52
53	12.57920	11.67618	10.86957	10.14057	9.49273	53
54	12.23778	11.38112	10.61151	9.91729	9.28770	54
55	11.89164	11.08052	10.34946	9.68778	9.09632	55
56	11.54089	10.77451	10.08150	9.45203	8.88860	56
57	11.18614	10.46358	9.80802	9.21037	8.67487	57
58	10.82780	10.14814	9.52927	8.96305	8.45532	58
59	10.46489	9.82864	9.24576	8.71030	8.23009	59
60	10.10687	9.50567	8.95795	8.45767	7.99966	60

TABLE III—*Continued*

Age	1 Life	2 Lives	3 Lives	4 Lives	5 Lives	Age
61	9.74148	9.17955	8.66609	8.19581	7.76415	61
62	9.37520	8.85124	8.37107	7.93008	7.52426	62
63	9.00833	8.52096	8.07306	7.66059	7.27997	63
64	8.64169	8.18957	7.77287	7.38808	7.03210	64
65	8.27598	7.85764	7.47105	7.11301	6.78093	65
66	7.91213	7.52614	7.16848	6.83623	6.52735	66
67	7.55060	7.19543	6.86543	6.55804	6.27159	67
68	7.19221	6.86635	6.56280	6.27920	6.01434	68
69	6.83799	6.53987	6.26142	6.00060	5.75640	69
70	6.48835	6.21646	5.96185	5.72270	5.49824	70
71	6.14434	5.89711	5.66501	5.44642	5.24076	71
72	5.80655	5.58243	5.37155	5.17236	4.98452	72
73	5.47540	5.27293	5.08199	4.90106	4.73013	73
74	5.15186	4.96954	4.79719	4.63342	4.47836	74
75	4.83667	4.67306	4.51801	4.37026	4.23008	75
76	4.53028	4.38393	4.24496	4.11209	3.98578	76
77	4.23303	4.10256	3.97849	3.85938	3.74599	77
78	3.94587	3.82995	3.71956	3.61316	3.51172	78
79	3.66868	3.56606	3.46826	3.37350	3.28310	79
80	3.40222	3.31169	3.22541	3.14126	3.06099	80
81	3.14668	3.06708	2.99130	2.91679	2.84580	81
82	2.90228	2.83257	2.76539	2.70053	2.63797	82
83	2.66926	2.60845	2.54970	2.49287	2.43795	83
84	2.44773	2.39484	2.34364	2.29405	2.24601	84
85	2.23714	2.19131	2.14688	2.10376	2.06192	85
86	2.03847	1.99892	1.96048	1.92315	1.88684	86
87	1.85114	1.81713	1.78402	1.75181	1.72043	87
88	1.67509	1.64596	1.61756	1.58987	1.56288	88
89	1.51041	1.48557	1.46129	1.43759	1.41446	89
90	1.35619	1.33508	1.31444	1.29424	1.27449	90
91	1.21321	1.19536	1.17788	1.16076	1.14399	91
92	1.07823	1.06327	1.04059	1.03419	1.02007	92
93	.95036	.93795	.92576	.91379	.90203	93
94	.83039	.82026	.81031	.80052	.79087	94
95	.70409	.69613	.68832	.68063	.67303	95
96	.56542	.55952	.55378	.54811	.54251	96
97	.43621	.43227	.42838	.42452	.42069	97
98	.24155	.23972	.23791	.23611	.23432	98

APPENDIX

In the foregoing remarks I have purposely omitted reference to the momently rate of mortality and its employment in determining the single age. The determination of Makeham's formula rests on yearly changes in the age, or a difference of unity in the variable, and it seemed to me that it would be introducing a possible confusion to intermingle differences and differentials.

To those interested in the use of the momently rate of mortality,

I may remind them that this function may be used in place of the powers of c by means of the relation

$$\mu_u = \mu_x + \mu_y + \mu_z + \ \ldots\ + \overline{m-1}\ \log s$$

remembering that the logarithm is Napierian and negative. For the Makehamized American table

$$\text{Nap. log } s = -.0075913053$$

The relation is established as follows:

$$\begin{aligned}\mu_x &= -\frac{d}{d_x}\log l_x = \frac{d}{d_x}\{\log k + x \log s + c^x \log g\}\\ &= -\log s - \log c \cdot \log g \cdot c^x\\ &= A + Bc^x\end{aligned}$$

Then $\mu_x + u_y + \mu_z = 3A + B\ (c^x + c^y + c^z)$

also $\mu_u = A + B\ c^u$

Hence $\mu_u = (\mu_x + \mu_y + \mu_z) - 2\ A$

$= (\mu_x + \mu_y + \mu_z) + 2 \log s$

and generally for m lives

$$\mu_u = (\mu_x + \mu_y + \ \ldots\ ) + \overline{m-1} \log s$$

For $m-1$	$\overline{m-1}\ \log s = 0.000$
2	-0.00759
3	-0.01518
4	-0.02277
5	-0.03037

Thus for 3 ages 33, 37, and 43:

	c Method	μ Method
33	32.45	.00863
37	49.48	.00918
43	93.16	.01058
		.02839
		.01518
	175.09 =	49 = .01321

TABLE OF MOMENTLY RATE OF MORTALITY

Age	Rate	Age	Rate	Age	Rate	Age	Rate
10.....	.00768	35....	.00888	55....	.01818	80....	.15540
11.....	.00769	36....	.00902	56....	.01936	81....	.17183
12.....	.00770	37....	.00918	57....	.02066	82....	.19010
13.....	.00772	38....	.00935	58....	.02212	83....	.21040
14.....	.00773	39....	.00955	59....	.02373	84....	.23295
15.....	.00775	40....	.00977	60....	.02553	85....	.25801
16.....	.00776	41....	.01001	61....	.02752	86....	.28586
17.....	.00778	42....	.01028	62....	.02974	87....	.31681
18.....	.00781	43....	.01058	63....	.03220	88....	.35120
19.....	.00783	44....	.01091	64....	.03494	89....	.38941
20.....	.00786	45....	.01128	65....	.03798	90....	.43187
21.....	.00788	46....	.01169	66....	.04136	91....	.47905
22.....	.00792	47....	.01215	67....	.04512	92....	.53149
23.....	.00795	48....	.01265	68....	.04929	93....	.58975
24.....	.00799	49....	.01321	69....	.05393	94....	.65449
25.....	.00804	50....	.01384	70....	.05908	95....	.72643
26.....	.00809	51....	.01453	71....	.06481	96....	.80637
27.....	.00814	52....	.01531	72....	.07117	97....	.89521
28.....	.00821	53....	.01617	73....	.07824	98....	.99392
29.....	.00827	54....	.01712	74....	.08610		
30.....	.00835			75....	.09483		
31.....	.00843			76....	.10453		
32.....	.00853			77....	.11531		
33.....	.00863			78....	.12729		
34.....	.00875			79....	.14060		

CALCULATION OF μ_x

$$\mu_x = -\log s - \tfrac{1}{2}\log g \log c\,\tfrac{1}{2}\, c^x$$

The initial value of μ_{10} is calculated, taking Nap. logs $\cdot\ \mu_{10} = .007683$

$$\begin{aligned} \text{Then}\quad \Delta\mu_x &= -(\lg g \cdot \log c)c^x(c-1) \\ \lg_{10}\Delta\mu &= \lg_{10} m + x \log c \\ \lg\ \Delta\mu_{10} &= \lg m + 10 \log c \\ \lg\ \Delta\mu_{11} &= \lg m + 10 \log c + \log c \end{aligned}$$

by continuous additions of $\log c = .045796$

x	$\log \Delta \mu_x$	$\Delta \mu_x$	μ_x
10	5.010072	.000010	.007683
11	.055868	11	.007693
12	.101664	13	.007704
13	.147460	14	.007717
14	.193256	16	.007731
.	.	.	.
.	.	.	.
.	.	.	.

PART II

In connection with the foregoing application of the Makehamized table it is interesting to note the changes in the annuity values from those deduced from the original table. The differences are practically negligible, as the following comparison shows:

ANNUITY VALUES

Age	Original Table	Makehamized Table	Age	Original Table	Makehamized Table
10.....	21.22	21.21	55.....	11.83	11.89
15.....	20.73	20.73	60.....	10.03	10.11
20.....	20.14	20.15	65.....	8.22	8.27
25.....	19.44	19.45	70.....	6.48	6.49
30.....	18.60	18.61	75.....	4.90	4.84
35.....	17.61	17.61	80.....	3.43	3.40
40.....	16.45	16.44	85.....	2.08	2.24
45.....	15.09	15.09	90.....	.87	1.36
50.....	13.53	13.57	95.....	.00	.70

It therefore becomes of much interest to trace the method by which it has been possible to evolve a formula which is so tractable and which at the same time reproduces so faithfully the original data. Now in examining any table giving the numbers recorded to be living at successive ages, it is apparent to even the casual observer that there exists a progression of some kind, either in the numbers living or perhaps more noticeable in the numbers of those dying. It follows naturally to seek the nature of this change, guided by the actual figures. There may exist at the same time some other indication of the nature of the progression, such as the age at which all lives pass out of observation.

The simplest expression for this kind of change is a decrease by equal amount; that is, an equal number of deaths at each age.

It is a simple transition, in thought, having once conceived the idea of a change proceeding in some regular way, from one form of change to another either equally simple or slightly more complex. Thus the idea of a geometrical series for the numbers living or for the numbers dying readily presents itself in the form of an increase such as that presented by the operation of interest, where the number living or dying at one age presents a percentage increase on the number living or dying at the preceding age. Or again the deaths may present themselves in a series wherein each term is greater than the preceding by the same constant number, an arithmetical progression.

Now these ideas are formed always, it must be remembered, with actual tables as a basis of observation; and it may very well happen that the actual numbers will not present any readily grasped sequence. We are led, or rather driven, then, to seek some other set of numbers, easily derived from the first, which, upon examination, may incline to those characteristics which other observed phenomena of nature are known to possess. In this search we are partly influenced by our previous knowledge of logarithms to suggest to ourselves that although the original numbers are unmanageable, their logarithms may not be so.

Thus step by step we proceed from the examination of the original figures to that of some simple function of them, such as their logarithms, or their logarithms multiplied by a constant, or their ratios, or the logarithms of their ratios. By this and other considerations Makeham conceived the idea that the logarithm of the probabilities of living multiplied by a constant were in geometrical progression. This idea follows naturally from the previous ideas I have outlined, as it is a transition from the sequence of the numbers living to simple functions of them, and suggested, moreover, by the numbers themselves after trials of simpler arrangements. So far, however, no attempt is made to give mathematical expression to these trials. The process of translating the conceptions, formed as I have outlined, into the language of mathematics introduces that branch wherein we deal with functions, known or unknown, of the variable under consideration. In the case in hand, the variable is the age and the function will be variously the number living, the number dying, the probabilities, or their ratios, or their logarithms, or the logarithms of these ratios. The idea formed is then expressed as an equation

connecting functions of successive values of the variable, these values being equidistant but not necessarily corresponding to unit differences of the variables. These differences being finite, the functions will correspond to equidistant finite values of the age, and the equations formed will consist of functions of finite values of the age combined with constant terms or constant factors, or both. Then by means of known principles of the calculus of finite differences, solutions are sought for the difference equations which present themselves.

In elucidation of this method I present a few of the more striking problems that offer themselves for consideration, together with the corresponding equations and their solution. It is, of course, not within the range of this paper to include the actual methods of solution, yet I may say that all the solutions here considered may be effected without difficulty and all the equations are linear; that is to say, of the first degree though not all of the first order. I have been obliged necessarily to suppose a knowledge of the solution of linear difference equations, and trust that I may have excited sufficient interest to induce my fellow-students to take up that study in cases where it has not been already mastered. Indeed I would take this opportunity to press upon our body the advisability of incorporating this branch of mathematics in the syllabus of examinations for our fellowship.

PROBLEMS

	Relation	Equation of Differences
1.	The number of deaths at each successive age is constant.	$u_{x+1}-u_x=a$ $u_x=ax+b$
2.	The number of deaths form a series in arithmetical progression.	$u_{x+2}-2u_{x+1}+u_x=a$ $u_x=ax^{(2)}+bx+c$
3.	The numbers living form a geometrical series.	$u_{x+1}-au_x=0$ $u_x=ca^x$
4.	The number of deaths form a series in geometrical progression.	$\Delta u_{x+1}-c\Delta u_x=0$ $u_x=a+bc^x$
5.	The probabilities of living form a geometrical series.	$u_{x+2}/u_{x+1}=mu_{x+1}/u_x$ $u_x=ks^xg^{x^2}$
6.	The logarithms of the numbers living are equidistant.	$\log u_{x+1}-\log u_x=\log g$ $u_x=kg^x$

Relation	Equation of Differences
7. The logarithms of the probabilities of living form a series with constant differences.	$\log u_{x+2} - 2\log u_{x+1} + \log u_x = m$ $u_x = ks^x g^{x^2}$ (*Vide* No. 5)
8. The logarithms of the probabilities of living are in geometrical progression.	$\log u_{x+2} - (c+1)\log u_{x+1} + c + \lg u_x = 0$ $u_x = kg^{c^x}$
9. The chances of living a year multiplied by a constant factor form a series whose logarithms are in geometrical progression.	$\lg u_{x+2} - (c+1)\log u_{x+1} + c \log u_x = (c-1)\lg a$ $u_x = ks^x g^{c^x}$

I will now add a few notes to some of these problems and their solutions.

The first, in which the number of deaths is constant, is interesting historically, as being the form of DeMoivre's hypothesis. He assumed, moreover, that all would die by attained age 86, and that of 86 persons one would die every year. If n be the complement of life at age x, the assumption leads to the expectation at any age being one-half the complement of life, and to the following expression for the annuity value:

$$a_x = \frac{n-(1+i)\,a_n}{ni} = \frac{2e-(1+i)\,a_{2e}}{2ie}$$

This formula gives nearly correct results with the Northampton Table and it is curious to mention that at age 43 with the American Table, the formula gives the absolutely correct expectation, 26.00 years; and 15.03 for the annuity value, the correct value being 15.65. In forming the equations connecting the functions according to the various assumptions it will be of assistance to take u as the leading function and form the equation in that letter. Thus in the present case the equation is $u_{x+1} - u_x = a$ and its solution $u_x = a_x + b$. If the deaths form a series in arithmetical progression with a common difference a, the equation will be found to be as given in my list. It is to be noticed that this and the preceding problem belong to the same family; in the first, the deaths or first differences of the 1 column are constant; in the second the second differences of the 1 column are constant; in the first the form of the function includes the first factorial of x; the second includes the second factorial of x; and, if we supposed that the differences of the d column formed an arithmetical pro-

gression, we should find that the form of the function would include the third factorial of x. Thus, the differences being those of the 1 column:

1st differences constant, $l_x = ax + b$
2d " " $l_x = ax^{(2)} + bx + c$
3d " " $l_x = ax^{(3)} + bx^{(2)} + cx + d$*

In passing to the problems wherein the progression is geometrical, the variable will be found to be involved as an exponent. Considering the simple case in which the numbers living form such a series, the equation is $u_{x+1} - au_x = 0$, the solution of which is $u_x = ka^x$. This assumption leads, also, to peculiar results, one of which is that no assignable limit can be placed on human life, and another, that the annuity values would be equal at all ages. The first proposition can be shown very readily by the expectation reducing to $a/a - 1$, an expression independemt of the age. The annuity value reduces to $va/1 - va$, which is also independent of the age. I will not dwell upon the other simple cases of my list, but pass on to those leading up to the forms of Gompertz and Makeham, involving cases in which the logarithm of some function of the numbers living evince some progression. We may take the logarithm of the numbers themselves, or the logarithms of their ratios, or the logarithms of the probabilities of living.

Thus $\log l_{x+1} - \log l_x = m$, or $\log \frac{l_{x+2}}{l_{x+1}} - \log \frac{l_{x+1}}{l_x} = m$.

In the first case $\log \frac{l_{x+1}}{l_x} = m$ and $\frac{l_{x+1}}{l_x} = e^m$, which reduces to a case already considered.

In the second case $\frac{l_{x+2} \cdot l_x}{(l_{x+1})^2} = e^m$, which also reduces to a case already considered.

Consider now that from a table of the numbers living the column is formed showing the probabilities of living one year at each age. Suppose that it is observed that the logarithms of these probabilities are practically in geometrical progression. What is the form of the function of the age, which represents the number living?

If p_x be the probability of living a year, then the problem

* Where $x^{(n)} = x(x-1)(x-2) \ldots (x-n+1)$.

gives $\log p_{x+1} = a \log p_x$, and substituting the values of the probabilities in terms of the numbers living, the equation becomes $\log l_{x+2} - \log l_{x+1} = c \log l_{x+1} - c \log l_x$, which is of the form $u_{x+2} - (c-1)u_{x+1} + cu_x = 0$.

This equation will give at once the solution $\log l_x = \frac{m}{c-1} c^x + c_1$ whence $l_x = kg^{c^x}$ by changing the constants. Otherwise the solution may be obtained more gradually by first solving the equation $\log p_{x+1} = a \log p_x$, giving $\log p_x = mc^x$, and then solving the equation $u_{x+1} - u_x = mc^x$. This is Gompertz' formula, and it is probable that he arrived at his conclusion in the manner I have here outlined, although in his paper he adopts an entirely different line of thought, arriving at his result through the expression for the force of mortality and a consideration of other natural phenomena. It is now an easy transition to the formula of Makeham.

After discussing the formula of Gompertz, Makeham proposed his variation in the following words: "To find an expression for the number living at any age when the law of mortality is such that the chances of living a year, multiplied by a constant factor, form a series whose logarithms are in geometrical progression."

There are several methods of handling this problem; it is, however, entirely in line with the previous propositions in this paper. The gradual way, of considering first the function that first presents itself, is probably the best as showing the forms of the various functions involved.

Thus the equations will be:

1. $\log (ap_{x+1}) - q \log (ap_x) = 0$

which gives

$$\log p_x = cq^x - \log a$$

2. $\log l_{x+1} - \log l_x = cq^x - \log a$

which gives

$$\log l_x = \frac{c}{q-1} q^x - x \log a + c_1$$

3. Finally $l_x = e^{c_1 + \frac{c}{q-1} q^x - x \log a}$

which gives

$$l_x = ks^x g^{q^x}$$

by changing the constants.

It is interesting to notice that there exists a very simple connection between the proposition of Gompertz and Makeham. The law of Gompertz reduces to "the logarithms of the probabilities of living any period are in geometrical progression" and that of Makeham reduces to the proposition that the first differences of these are in goemetrical progression. Taking this view of Makeham's law, it will be found that the resulting equation is much more simple and the form of the function 1 more easily obtained. Our equation of differences is then

$$\Delta \log p_{x+1} - q\Delta \log p_x = 0$$

giving an integration $\Delta \log p_x = cq^x$ whence $l_x = ks^x g^{q^x}$ as before.

This brings me to the point I had marked as a halting-place and the conclusion of this outline. The subject has for me a great attraction and in these notes I have endeavored to explain the course that has suggested itself to me as a progressive method of reaching the ideas that will lead to the results reached by men such as Gompertz, Makeham, and Thiele. At the same time I wish to attract the attention of our students to the benefits of a careful study of the calculus of finite differences apart from the differential and integral calculus. Dr. Boole expressed this idea when he wrote: "Such a method [the method of differential equations] should not be resorted to when the direct consideration of the operations properly belonging to the Difference-Calculus suffices to solve our problems."

DISCUSSION OF "FORMULA FOR CONTINUED INSTALMENTS" BY LUCIUS McADAM

(Vol. I, No. 2, page 1)

By H. L. RIETZ

In the beginning of the paper before us, attention is called to the facts that the formula for continued instalments is published in the *System and Tables of Life Insurance* by Levi Meech, and that a derivation of the formula was published by Mr. Hugh W. Robertson in the *Transactions of the Actuarial Society of America*, Vol. VII. In the present paper, Mr. McAdam gives an alternate derivation of the same formula.

It may be worth while in this discussion to consider the starting-points of the two derivations as they differ in this respect. The fixed instalments being looked upon in the same way in the two derivations, we give attention only to the continued instalments that follow the t instalments of fixed period.

From the viewpoint of Mr. Robertson's derivation, the present value of an instalment of 1, at time $n+t$ after the date of insurance, is the present value of the expectation of 1 contingent on the condition that the beneficiary y be living at age $y+n+t$ and that the insured x shall have died t years earlier. This gives

$$V^{n+t}{}_{n+t}\, p_y(1-{}_np_x)\,,$$

and the sum of such expressions with respect to n,

$$\Sigma_{n=1}^{n=\infty} V^{n+t}{}_{n+t}\, p_y(1-{}_np_x)\,,$$

gives the total extra single net premium for the continued instalments.

From the viewpoint of Mr. McAdam's derivation, the extra net single premium for the continued instalments is equal to the present value of a life annuity to the beneficiary y deferred t years diminished by the present value of the expectation of 1 per year contingent on the condition that y be living at age $y+n+t$ and that x be living at age $x+n$ when n takes values from 1 to ∞. With these two views as starting-points, the same resulting formula is obtained by simple algebraic transformations.

Mr. McAdam calls attention to the fact that, by strict compu-

tation, the reserve on the extra premiums for continued instalments is frequently negative. This question is treated to some extent by Mr. Welch in the *Transactions of the Actuarial Society of America*, Vol. VII, p. 371, with the approximate general result that unless the beneficiary is less than 10 years younger than the insured, negative reserves may be looked for during the early years if not during the whole term of the contract.

Mr. Welch further indicates that these negative reserves are specially interesting because in giving surrender values for these continued instalment policies, the values of the regular twenty instalment policy are usually allowed, no account whatever being taken of the contingent deferred annuity provided for in the contract. We are thus brought, under the continued instalment plan, with certain combinations of ages, to the fact that a company may give to retiring policy holder more than the policy has accumulated.

Mr. McAdam states that this is not a safe proposition to handle, and suggests a sort of common-sense method of computing the entire reserve as a proportion of the reserve on the original policy. The principles involved in this method are not clear to me but I shall not attempt to criticize the method as I have no substitute to offer for it. I think, however, it would be of value to further consider this question.

DISCUSSION OF MR. McADAM'S PAPER

By W. H. GOULD

In a paper which merely gives an actuarial formula, there is very little scope for discussion, as the paper then becomes exact and mathematical.

I would like to draw attention to what is evidently a typographical error, on the second page of the paper, in which "A" has been substituted for "a."

The formula in question as presented by Mr. McAdam I have used for a number of years, together with the corresponding formula when the continuous instalment benefit is applied to endowment insurance, providing for the payment of the benefit after the fixed term of years (t) to the survivor of the insured and beneficiary.

Mr. Robertson, in his paper (Vol. VII, No. 27, *Transactions Actuarial Society*) to which Mr. McAdam refers, gives this benefit deduced in his own manner. A formula similar to the one in the paper before us can be deduced, to my mind, very simply. Let us consider the benefit as presented in the paper to be payable only if the insured shall die within the endowment period, then the formula becomes merely that for temporary annuities for the period of $(m-1)$ years, "m" being the endowment period, and the formula becomes:

$$\frac{D_{y+t}}{D_y}(a_{y+t}\,m-1-a_{y+t:x}\,m-1)$$

For that portion of the benefit which applies to the continuous instalment, should the insured survive to mature the endowment, we have a survivorship annuity deferred $(t+m-1)$ years, or, this survivorship annuity by resolving it into its component parts, becomes:

$$\frac{D_{y+t+m-1}}{D_y}a_{y+t+m-1}+\frac{D_{x+t+m-1}}{D_x}a_{x+t+m-1}-\frac{{}^{t+m-1}D_{xy}}{D_{xy}}\times{}^{t+m-1}a_{xy}$$

So that the value of the continuous instalment benefit applied to endowment insurance, is:

$$\frac{D_{y+t}}{D_y}(a_{y+t}-a_{y+t:x}\;\;m-1)+\frac{{}^{t+m-1}D_x}{D_x}\cdot{}^{t+m-1}a_x$$
$$-\frac{{}^{t+m-1}D_{xy}}{D_{xy}}\cdot{}^{t+m-1}a_{xy}$$

which can be converted into this formula for the single premium:

$$\frac{D_{y+t}}{D_y}(a_{y+t}-a_{y+t:\,x\;\;m-1})+\frac{{}^{\wedge}N_{x+t+m}}{D_x}-\frac{{}^{\wedge}N_{x+t+m\,:\,y+t+m}}{D_{xy}}$$

The reasoning by which Mr. Robertson arrives at his formula is somewhat different from this, and I would suggest that reference be made to his paper.

The warning given in the paper regarding reserve on continuous instalment portion being frequently negative, should be observed.

Another suggestion which gives a safe reserve in the case of valuation of the benefit previous to the policy becoming a claim, is to merely accumulate the net annual premium at interest, ignoring the mortality. The reserve for the benefit after the policy becomes a claim resolves itself into the valuation of the annuities involved.

ORAL DISCUSSION OF MR. McADAM'S PAPER

By H. W. BUTTOLPH

I am not prepared to discuss at any length Mr. McAdam's paper but there is just one suggestion that I would like to make.

Mr. McAdam warns against the dangers arising from negative reserves. It seems to me that this warning should be extended to cover the premiums which result in negative reserves. I believe that in practice it would be found that a majority of the policies actually issued under this plan, if this continued instalment be viewed separately, would result in creating negative reserves.

In handling this matter the practical question arises as to what ruling insurance departments are going to make as regards the surrender values required. I believe that those states which have non-forfeiture laws similar to that of Illinois, Indiana, Michigan, Ohio, and others will hold that the same minimum values must be granted under this form of policy as would be required in the case of an instalment policy under which the instalments are payable for the same length of time as that covered by the instalments which are certainly payable under the form which we are now discussing. In other words, as far as the surrender values are concerned, no consideration will be given to the continued instalments, whether this feature would result in a positive or a negative reserve. That being the case, it becomes necessary to eliminate in practice all negative reserves, and this can be done only by charging an annual premium high enough to cover the expected cost for the first year.

If am correct in my belief that a majority of the policies issued would be of this character, it would seem as though Mr. McAdam's suggestion in regard to a convenient method for calculating the reserve under this particular feature would result in the accumulation of an entirely unnecessary reserve in the case of live policies and would not leave the company the proper funds with which to cover the liability which would be charged against it on account of such policies as had become claims.

DISCUSSION OF "FIRST YEAR RESERVES: SOME CRITICISMS AND A SUGGESTION" BY E. W. HYDE

(Vol. I, No. 2, page 5)

By J. H. NITCHIE

What is now called the preliminary term method of valuation is the outgrowth of a suggestion forty or fifty years ago, that in the valuation of policies notice be taken of the fact that the first year's income is largely or wholly exhausted in the procuring of business. This suggestion received favorable and creditable endorsement even before it came into practical use. Those who commended it probably had little thought of applying it except to the Ordinary Life plan. Its use in connection with other plans was followed by its restriction to the Ordinary Life plan or its modification so that for other plans the amount thus capitalized for initial expenses would in all plans approximate that for the Ordinary Life plan at the same age of issue. This interpretation of the modified plan seems to me susceptible of reasonable explanation even to the non-technical inquirer.

A similar form of modification was proposed, extending the full preliminary term basis to the 20 Payment Life plan and modifying the higher rate plans on the 20 Payment Life basis.

These two methods, with variations, have been adopted for government valuations.

The limitation of expense provisions being the object sought in the proposals for modification, the proportions of the respective premiums and reserves were a secondary consideration—were possibly not at all considered.

Professor Hyde proposes a system which would produce greater uniformity in these features, and I think that he has presented a very ingenious and attractive method. If we had not already launched so many and varying systems of valuation, it might be very desirable to consider the adoption of a plan based on his suggestions. My fear is that the result of such attempts at the present time would be greater confusion and deeper chaos.

It would be difficult to bring about agreement among state and company officials as to the values of the proposed "k" function; then too the technical tables would be as varied as the values of this function.

The Ordinary Life plan, under preliminary term valuation, shows some of the incongruities, which Professor Hyde criticizes in the disproportion between the first year and renewal premiums at different ages. He suggests a modification of that plan according to the proportional method and calls attention to the negative reserves resulting from using the particular value of "k" which he employs. This could be obviated by adopting the proportions indicated by the premiums for the earliest age, say 20 or even 15, and using the same "k" for all higher ages; thus the first year's reserve for the initial age would be zero and for all subsequent ages would be positive.

The incongruities which he has pointed out in the modified system are partly due to those in the Ordinary Life Preliminary Term plan. If this plan were to be proportioned as above indicated and if the modified preliminary term plans were built up on the resulting net premiums, the range of fluctuations would be greatly reduced.

It will be noticed that the 20 Payment Life Preliminary Term premiums, up to age 55, exhibit proportions which approach uniformity, and that the premiums for other plans, modified on the 20 Payment Life basis, also show a narrower range of fluctuations than those criticized by Professor Hyde.

I cannot see that the formulae for the modified preliminary term plans are more complicated than those which he proposes; the latter seem to introduce complications which would militate against the adoption of the desirable features of his system.

I think that arbitrary values of "k," or the application of this function, as above, to the fundamental plan only, would be preferable to these elaborate formulae. Professor Hyde suggests that the functions proposed by him may be inapplicable to certain shorter terms, and that in such cases the level premium system be substituted: but the usual methods of modification apply consistently to all these forms.

If uniformity in these proportions is more desirable, Professor Hyde's method offers it, but if the uniformity in expense provisions, which was sought in the modified systems, is rather to be desired, we have it in the modified preliminary term plans. It is difficult to reconcile both features in one method, except so far as the fundamental plan might be modified, on the lines of Professor Hyde's method.

DISCUSSION OF "GROUP VALUATION" BY R. M. WEBB

(Vol. I, No. 2, page 14)

By CHARLES H. BECKETT

This paper belongs to a class of contributions prepared to assist the actuary in the somewhat tedious labor of preparing terminal and mean reserve valuations for the company, and does not promulgate a new theory.

The writer explains with sufficient detail his manner of grouping the policies by attained ages, the chief advantage as stated being that the number of groups is thereby reduced to a minimum.

Taking the Actuaries Combined 4 per cent Preliminary Term as a basis, and the Ordinary Whole Life Premium as a unit, the premium differences multiplied by N_{x+1} are tabulated for all ages from 20 to 65 inclusive.

The work of computing the terminal reserve readily reduces to eight simple operations with the calculating machine, and is confined to one group for attained ages which includes all regular forms of policies within the premium-paying period. By substituting the mean values for the three factors in the multiplication the mean reserves are likewise taken off.

The society is indebted to the author for a clear and full explanation of his method of preparing the work sheets from the office records.

My interest in the paper led me to prepare like tables upon American $3\frac{1}{2}$ per cent Modified to 19 Payment Life, and I have found the suggestions valuable in preparing work sheets. For convenience I used 100,000 as a constant multiplier and divisor instead of D_{40}.

The purpose of valuing a company's business is usually twofold:

First: To obtain the information required in the uniform report blank including the gain and loss exhibit.

Second: For internal purposes and the study of classes of policies and years of exposure.

The books should therefore be designed so as to furnish all desired information most conveniently and with the least amount of duplication. I think the terminal and mean reserves upon policies which do not provide for varying net premiums can be

found with less work and in shorter time by grouping by attained ages than by any other system. These latter forms seem to be increasing rapidly and I think all valuations pertaining to them can be ascertained most practically by separate groups as per age and year of entry.

Semi-annual, quarterly, or even monthly reports are frequently desired by the board. By adopting the plan of keeping the age groups correct from month to month, by a system of an on-and-off record, an intervaluation could be quickly made.

I think that any young company should be ready to furnish such a report upon short notice.

It appears to me that, while the British plan has the advantage of reducing the groups, the more elaborate plan of a seriatim valuation, which is made preferably from day to day, combined with a group system arranged as to kind, year, date, and age of issue, is worth all its costs in clerical work. However there is likely to be a difference of opinion as to the desirability of any seriatim valuation.

With modern office equipment including first-class adding and computing machines, it would seem to be no longer necessary to make two separate valuations merely for verification purposes.

Whenever the Insurance Department of the home state also values a company's business, it is convenient to use the same group method, and age groups would serve very well if used by both, but for the second purpose, viz., internal use, it would seem that a grouping by the most common practice among American companies, by age, kind, and year of entry, is highly desirable.

DISCUSSION OF "GROUP VALUATION" BY R. M. WEBB

(Vol. I, No. 2, page 14)

By J. CHARLES SEITZ

If a life insurance office makes only one valuation as of December 31 in each calendar year, the group method of valuation, as used by Mr. Webb, will, no doubt, be found convenient. I prefer a method of valuation such that the company will have knowledge of its reserve liability at the end of each month. Insurance companies will have elaborate plans and spare no expense in giving details of work in their accounting department and in their agency department, but the actuarial department is too often prone to do its work in aggregate in order to save time and expense. It is as important to know the trend of the policy liability as it is the trend of the income and disbursements. On account of my preference for a valuation for monthly results, I am not enthusiastic over the group method of valuation, as it would not readily give the monthly results of the policy liability. I therefore consider Mr. Webb's suggestion as to valuation purely in the light of his remarks that it may be used as a check valuation. All of us are pleased to have some second method of proving the calculations, and since the group method is on such entirely different lines from the seriatim, or modification of the seriatim method, it is an excellent system for checking results. On the other hand, with the development of the insurance department, is it at all necessary to have any check on the company's valuation? The insurance department furnishes its valuation and usually applies the title of correctness to such valuation in preference to the company's valuation if the differences are not great. They have all the data concerning the policies and the company could therefore eliminate the trouble of the check valuation of its own and allow the insurance department's computations to prove the company's correctness of computation.

The group method of valuation should be of great assistance to the insurance departments. Either the insurance department or the company would have to compute the valuation factors for the various reserve methods under which the policies of the various companies have been written. If the companies are using a group valuation to a great extent it would be well to have the

valuation factors and other tabulation under the various reserve methods computed and printed in *The Record* for permanent use. The Institute would be benefited if Mr. Webb was disposed to publish in *The Record* his proposed valuation factors and other data for the Illinois standard of reserve, the computation of which he mentions he was about to make.

I recognize the fact that familiarity with any system makes it very easy to the one that uses it, but I believe Mr. Webb's discussion of his actual use of the group valuation appears less troublesome than it really is. The clerk must handle once each policy card in order to obtain the data, since the amount of insurance must be placed on the square-ruled sheet shown on page 19. If the policy records are filed by plans of policies, then one such square-ruled sheet will be before the clerk and it will be used to insert the amount of insurance. By this plan, when there is more than one policy for the same square, then erasures must be made and a new total entered. This would be a prolific chance for error besides taking much extra time in the erasures.

The results of the group valuation are in excellent shape for the computation of cost of insurance, since the amount of insurance and the reserve at each attained age are given, from which data the amount of risk may be multiplied by the probability of the insured dying. Having the results at the end of each year, the cost of insurance may then be computed on the basis of the article by H. N. Shephard upon page 219 of the second volume of the *Transactions of the Fifth International Congress of Actuaries.*

Mr. Shephard's literal explanation of his formula is as follows: "Classify the policies by ages attained (done most easily by means of the office year of birth, i.e., year of entry less age at entry) and obtain the totals of the sum assured and of the mean reserve at the end of a calendar year for all of the same attained age; multiply by one-half of the mean force of mortality at the end of the calendar year, and add corresponding figures for end of previous year, and we obtain expected claims and reserves expected to be released by death during the calendar year, the difference being the expected death losses."

In connection with my views as to the valuation for monthly results, it might interest you to know of the plan which I have in use. Every office will have the terminal and mean reserves for each of its policies. With the table of mean reserves, the most

ordinary clerk would be able to properly make the valuations. Upon the reserve card are placed the data concerning each policy with space upon the front side for the mean reserve for twenty-five years. The cards are filed under the kind of policy, sub-filed by year of issue, and further sub-filed by age. The card has a monthly tab at the top for denoting the anniversary month of the policy. The guide card at each age group has a ruled schedule upon which is inserted the mean reserves for $1,000 of insurance. Each month, those cards which have a tab bearing that month have entered upon them the mean reserve as shown on the guide card preceding its age group. There is noted the increase between the totals of the old reserve and the new reserve, which, together with the mean reserve of new business, reinstated business, and ceased business, make up the increases and decreases for the month in the reserve account. The card at the end of the month will show the mean reserve for any policy of which that month is the middle of any twelve calendar months. Upon December 31 of each year it is assumed that all of the policies have been issued upon July 1, and to have a valuation as of July 1, it would not make a financial statement look very encouraging to place against each policy the initial reserve, that is, the terminal reserve plus the net premium due upon July 1. It is necessary to consider the end of each month as the December 31 of the twelve months preceding.

On the reverse side of the reserve card there is a schedule for obtaining the information in regard to the terminated policies for the gain and loss exhibit.

ORAL DISCUSSION OF MR. WEBB'S PAPER

By H. W. BUTTOLPH

There is one point in Mr. Webb's paper upon which I do not at all agree with him and to which I wish to take exception.

Mr. Webb assumes that the theory of a mean valuation is that all policies on the average are issued as of July 1. I do not believe that this is the proper assumption to be made in the support of the theory of mean valuation. It seems to me that the assumption that is in effect made is that, whatever the date of the valuation, the policies issued during any twelve-month period terminating at that time of the year will on the average be in the middle of their respective policy years. To assume that a terminal valuation would be the proper method of determining the company's liability on June 30 would in my opinion be extremely erroneous.

I think this can easily be illustrated. Suppose a company which began business January 1, 1910, desired a valuation as of June 30, 1911, the policies issued during the year 1910 being evenly distributed throughout the year and all issued upon the full preliminary term basis. It is evident that one-half of the policies issued during 1910 and in force on June 30, 1911, would have paid their first renewal premium and that the other half would still impose upon the company some liability on account of the unexpired term of insurance. Under the first half, the company would have to assume a very considerable liability, whereas, according to Mr. Webb's theory, it being assumed that all policies issued in 1910 were issued on the 1st day of July of that year, the year just having expired and all policies being valued upon a full preliminary term basis, there would be no liability charged against the company.

It will readily be seen that, while this first year's business perhaps is the extreme case, yet the same theory would hold good for business issued in any year.

QUESTIONS FOR SECTIONS A AND B OF THE ASSOCIATE EXAMINATION

SECTION A. MORNING SESSION

1. Divide 243 into three parts such that one-half of the first, one-third of the second, and one-fourth of the third part shall be equal to each other.

2. Solve the equations

$$y^2 - xy = 15$$
$$x^2 + xy = 14$$

3. Find $\sqrt[3]{127}$ correct to five decimal places by the Binomial Theorem.

4. Show that the series:

$$1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \ldots .$$

is convergent for all values of x less than 1, and divergent for all values of x greater than 1.

5. Find the sum of the series:

$$2 + x + 5x^2 + 7x^3 + 17x^4 + \ldots .$$

and show that the series is convergent for all values of x less than $\frac{1}{2}$.

6. Show that in the common system of logarithms, when the position of the decimal point in any number is changed, the logarithm of the number is changed in characteristic but not in mantissa.

7. Find the value of:

$$(.9662)^{-.3014}$$

using the logarithm of a logarithm in your calculation.

8. There are six companies in all in any of which an agent may place insurance. He has three term policies and three whole-life policies which are to be placed in such a way that not more than one term policy and two life policies shall be placed in any one company. In how many different ways may the policies be placed?

SECTION A. AFTERNOON SESSION

9. If ${}_np_x$ (or ${}_np_y$) is the probability that an individual aged x (or y) will live n years, find the probability that:

a) At least one of x and y will survive n years.

b) x will survive n years and y will die within n years.

c) At least one of the lives will fail within n years.

d) Both lives will fail in the nth year.

10. Given $\log 61 = 1.78533$, $\log 62 = 1.79239$, $\log 63 = 1.79934$, $\log 64 = 1.80618$; find $\log 63.527$ approximately by interpolation, making use of all four elements of the given data.

11. In a triangle A B C, the side A C is produced to D and the angles B A C and B C D are bisected by straight lines which meet in F. Show that they contain an angle equal to half the angle at B.

12. If from any point without a circle a tangent line and a secant line be drawn, then the rectangle contained by the whole secant line and the part of it without the circle is equal to the square on the tangent line. Prove.

13. Find the derivative (or differential coefficient) of

a) $(g)^{c^x}$ where g and c are constants

b) $(x^2-1)\sqrt{(x^3+1)}$

14. Evaluate:

a) $\dfrac{dx}{2x^2+3x+1}$

b) $\dfrac{(2x^3+1)dx}{(x+1)(x+2)}$

c) $x^m \log x\, dx$

15. State Taylor's Theorem and apply it to obtain a development of $\log e^x$ in the form of a power series in $x-1$.

SECTION B. MORNING SESSION

1. The insured obtained a policy loan of \$400 dated January 15, 1911, and due January 15, 1912. Interest was paid in advance at the rate of 6 per cent per annum, with right to receive unearned interest upon repayment of loan. Upon July 15, 1911, the insured remitted to the compay \$100, with instructions to have it applied to the loan. What is the outstanding amount of loan, and show

the required debit and credit entries in connection with the remittance.

2. (*a*) A 20-year 6 per cent bond for $1,000, interest payable semiannually, is bought fifteen years from date of issue. What is the price of the bond to net the purchaser 5 per cent payable semiannually?

b) Prepare an amortization schedule for above bond.

3. Find the probability expressed by ${}_{n-1}q_{xy}$ and give the interpretation of it.

4. Express the following in commutation symbols:

$$A_x;\ P_x;\ {}_nV_x;\ A_{x:n};\ A^1_{x\ n};\ {}_nAE_x;\ (\mathrm{IA})_x.$$

5. From the formula $A_x = v(1+a_x) - a_x$,
derive the formulas,

$$A_x = \frac{1 - ia_x}{1+i},$$

and

$$A_x = 1 - d\,(1+a_x)$$

and explain the meaning in each case.

6. Derive two methods of the computation of the Terminal Reserve at the end of the nth year of an Ordinary Life policy.

SECTION B. AFTERNOON SESSION

7. Show in notational form the first year's mean reserve of a Paid-up Term insurance of n years, beginning at age x.

8. State in detail the various steps and formulas you would use in calculating the Terminal Reserve on a 10-payment Life policy modified on the Ordinary Life basis.

9. (*a*) Given $q_{30} = .08429$, and ${}_1V_{30} = 10.49$, on a policy for $1,000, find the "Cost of Insurance" under this policy for the year.

b) Given $P_{30} = 17.19$; ${}_1V_{30} = 10.49$ and $i = .03$, on a policy for $1,000, find the "Cost of Insurance" under this policy for the year.

10. Given $a_x = 18.6054$ and $a_{x+n} = 15.0867$, find ${}_nV_x$.

11. Derive the formula for A_{xy} in terms of v and l. Discuss the two forms of commutation columns suggested for evaluation of such a formula.

12. A policy issued at age 35 provides that it will be automatically continued at the end of the fifth year as term insurance for five years 194 days, provided no indebtedness exists. There is, however, an indebtedness of $25.00 outstanding at the end of the fifth year. Explain the ways that the policy may treat this condition and give the final results of which the policy-holder should be advised. (Tables of Single Premiums of Temporary Insurance are furnished.)

SYNOPSIS OF THE MINUTES OF THE ANNUAL MEETING, JUNE 6 AND 7, 1911

The annual meeting of the American Institute of Actuaries was held on June 6 and 7, 1911, at the Hotel La Salle, Chicago.

The meeting was called to order at 10:30 A.M. by President Buttolph, and the following members responded to roll call:

FELLOWS

Anderson	Draper	Rietz
Appleby	Gould	Seitz
Arnold	Haight	Stookey
Beckett	Higdon	Vail (C. M.)
Buttolph	Hyde	Webb
Campbell	McAdam	Wilbur
Carr	Nitchie	Wynn
Carter	Nueske	

ASSOCIATES

Bishop	Stayman
Brimstin	Sullivan
Dark	Vail (R. S.)

CONTRIBUTING MEMBERS

Company	Representative
The Bankers Reserve Life	Bishop
Federal Life	Appleby; Brimstin
Illinois Life	Arnold; Stookey
Kansas City Life	Webb
National Life of U.S.A	Carter
Scranton Life	Napier (W. E.)
Reserve Loan	Stayman
Pittsburg Life and Trust	Dunlap (E. O.)
Security Life	Seitz
State Life	Wynn
U. S. Annuity and Life	McAdam

The President briefly addressed the members upon their bond of association—the Institute, his remarks being printed in full elsewhere in the *Record*.

The minutes of the last annual meeting and of the interim meetings of the Board of Governors were read and duly approved.

Librarian Carter submitted a report which showed that the growth of the Library was progressing rapidly. During the two

years' existence of the Library about 125 volumes and pamphlets have been secured, including complete sets of the *Journal of the Institute of Actuaries*, the *Transactions of the Actuarial Society of America*, the *Transactions of the International Congress of Actuaries*, and Walford's *Insurance Cyclopedia*.

The following are the rules of the Library:

Books in the Library may be consulted at any time by Fellows, Associates, and authorized representatives of Contributing Members.

Books may be withdrawn from the Library by any Member and retained for a period not exceeding two weeks. Such Member assumes responsibility for the return to the Library, in good condition, of any book withdrawn, and if books are to be sent by mail or express shall pay transportation charges.

A Committee for auditing the treasurer's accounts, a nominating Committee, and a Committee on Change of Date of Annual Meeting were appointed by the President, to report later.

The meeting adjourned at 12:15 P.M. and reconvened at 2:00 P.M.

Treasurer Campbell submitted his report, which showed a cash balance of $508.02. The report was approved on the recommendation of the Auditing Committee.

Mr. Draper stated that he was pleased to offer the use of his library to any members of the Institute under the rules governing the Institute Library. He urged that other members do likewise.

Mr. Dunlap, representative of the Pittsburg Life and Trust, stated that such a privilege was accorded to the members of the Institute in regard to the library of his Company. This Company has a library composed of about three thousand volumes, pamphlets, and manuscripts.

A motion was made by Mr. Draper that the Institute consider the expediency of determining the mortality of the first five years on the business of companies operating in the Middle and Western States.

Mr. McAdam read his paper as elsewhere printed in this number.

The morning session on June 7 began at 10:30 A.M.

The Nominating Committee made its report, following which report the officers and Board members were elected as shown in this number of the *Record*.

The Committee on Change of Date of Annual Meeting was con-

tinued in order to prepare the proposed amendment to the Constitution in order to provide for the change of date.

The Committee on Education made its report through Chairman Arnold, and by motion the Committee was continued as the Educational Committee for the purpose outlined in the report as follows:

> The Committee recommends that an Educational Committee be appointed from year to year with which students who are looking forward to taking the entrance examination for admission to the Institute may consult respecting the proper studies for them to pursue, and that this Committee shall be charged with the further duty of encouraging and promoting interest in such studies among possible candidates for admission to the Institute, and of interesting various institutions of learning in the insertion of such courses in their curricula where practicable, and encouraging general instruction in insurance in courses leading to the Bachelor Degree in all American colleges. The Committee suggests that this proposed Educational Committee outline a course of studies, suggesting the textbooks preferably to be used, and that the same be published for general distribution among prospective candidates.

Mr. Rietz read the paper as shown elsewhere in this number of the *Record*.

The discussion of the papers which appeared in No. 2 of the *Record* were made at this time and appear elsewhere in this number of the *Record*.

The meeting adjourned at 12:30 and reconvened at 2:30 P.M.

Mr. Webb opened the proceedings by reading his paper.

A vote of thanks was extended to the officers and members of the Board for their work during the past year.

A vote of thanks was extended to the Chicago members of the Institute for the attentions shown to the visiting members.

The meeting adjourned at 4:00 P.M.

CONSTITUTION

NAME

Art. I. This organization shall be called "American Institute of Actuaries."

OBJECT

Art. II. The object of the American Institute of Actuaries is to advance the Science of Insurance Mathematics by associating together persons of like interests for the presentation and discussion of papers and for the consideration of such other matters as may properly come before the Institute.

MEMBERSHIP

Art. III. The membership of the Institute shall be composed of three classes, Fellows, Associates, and Contributing Members.

OFFICERS

Art. IV. The Officers shall consist of a President, a Vice-President, a Secretary, a Treasurer, and a Librarian.

GOVERNORS

Art. V. The Board of Governors shall consist of the Officers, the ex-Presidents, and six other Fellows chosen as follows: Upon adoption of this constitution, six Governors shall be elected; two for three years, two for two years, and two for one year, and annually thereafter two shall be elected for three years. The Board of Governors shall have power to fill all vacancies occurring by death or resignation of Officers or Governors.

DUTIES OF OFFICERS AND GOVERNORS

Art. VI. The duties of the officers shall be such as usually appertain to their respective offices. The duties of the Board of Governors shall be in general to manage the affairs of the Institute. Appropriate subcommittees may be appointed by the Board of Governors for such matters as may be from time to time found desirable.

ELECTION OF OFFICERS AND GOVERNORS

Art. VII. The Officers shall be elected by ballot at the annual meeting for the term of one year. The Governors for three years. The ballots to be cast in person or by mail. The President or

Vice-President shall not be eligible for the same office for more than two consecutive years.

MEETINGS

Art. VIII. The annual meeting of the Institute shall be held on the first Tuesday of June in each year at such place as designated by the Board of Governors. Other meetings may be called by the Board of Governors from time to time, and shall be called by the President at any time upon the written request of five Fellows; twenty days' notice of such meetings shall be given by the Secretary.

ASSOCIATES

Art. IX. Any one of legal age, pursuing actuarial studies and looking forward to a future Fellowship, may on the conditions hereinafter stated be enrolled as an Associate and as such shall be entitled to all the privileges of the Institute except those of voting and holding office. The conditions are: that he shall be nominated by at least two Fellows of the Institute; that the notice of such nomination shall be given by the Secretary to each Fellow of the Institute; that not less than twenty days after such notice has been given, his nomination shall be approved by the Board of Governors; that he shall subsequently pay such examination fee and pass such preliminary examination as may be prescribed by the Board of Governors. When the Board of Governors finds that a candidate has passed an equivalent examination required by another recognized society of actuaries, it may waive the examination hereby required.

ADMISSION TO FELLOWSHIP

Art. X. Any Associate who has been such for at least ten months, and is twenty-five years of age, may apply to the Board of Governors for Fellowship. Every member of the Board of Governors shall be notified confidentially of such application at least twenty days before the meeting at which it shall be acted upon. If this application be approved by the Board of Governors he shall become a Fellow on paying such examination fee and passing such final examination as shall be prescribed by the Board of Governors. Otherwise no one shall be admitted as a Fellow unless recommended by a duly called meeting of the Board of Governors, followed by a ballot of the Institute either in person or by mail with not more than four negative votes and with affirmative votes not less than

half of the number of Fellows, twenty days' notice having been given to each Fellow.

CONTRIBUTING MEMBERS

Art. XI. A Contributing Member shall be any legal reserve Life Insurance Company that desires Membership in the Institute. Any person nominated by said Company shall have the privilege of attendance at any of the Annual or Special Meetings of the Institute (except executive sessions), also shall have the use, jointly with other members, of the Library, upon complying with the rules of the Institute governing same.

QUORUM

Art. XII. Four members of the Board of Governors shall constitute a quorum. A majority of the Fellows of the Institute at any duly called meeting shall constitute a quorum.

EXPULSION OR SUSPENSION OF MEMBERS

Art. XIII. The Board of Governors shall receive and hear all complaints preferred by any Fellow of this Institute against any other Fellow or Associate for misconduct in his relations to the Institute, or in his profession, or the practice thereof, or affecting the interests of the actuarial profession; provided such complaints shall be in writing plainly and specifically stating the matter complained of, and subscribed by the complainant.

The proceedings of the Board of Governors upon such complaint shall be deemed confidential and kept secret except so far as written or printed reports of the same shall be necessarily and officially made in the Institute.

Each party may appear personally and by counsel. Witnesses shall vouch for the truth of their statements on their word of honor. The Board of Governors shall make its decision, and if it finds the complaint to be true and of sufficient importance, it will so report to the institute with its recommendations.

The Institute will take such action on the report as it shall see fit; but no member shall be expelled or suspended unless by a vote of at least two-thirds of the members present and voting.

Art. XIV. No resolution expressive of opinion shall be entertained at any meeting.

AMENDMENTS

Art. XV. This Constitution may be amended by an affirmative vote of two-thirds of the Fellows present at any meeting held at least three months after notice of such proposed amendment shall have been sent to each Fellow by the Institute.

BY-LAWS

Art. I. *Order of Business.*—At a meeting of the Institute the following order of business shall be observed, unless the Institute votes otherwise for the time being:

1. Calling of the roll.
2. Address or remarks by the President.
3. Minutes of the last meeting.
4. Summary of the business transacted by the Board since the last meeting of the Institute, presented for its information and approval.
5. Candidates for Fellowship considered and voted upon.
6. Reports of Committees and Officers.
7. At annual meeting, election of Officers and Governors.
8. Unfinished business; new business.
9. Reading of papers prepared for presentation to the Institute.
10. Discussion of papers presented.

The Board of Governors' meetings shall proceed in this order as far as it may apply to them.

Art. II. *Meetings of the Board of Governors* shall be called whenever the President, or four members of the Board of Governors so request, but not without sending notice to each member of the Board ten or more days before the time appointed. Such notice shall state the subjects intended to be brought before the meeting; should other matters be passed upon, any member of the Board shall have the right to reopen the same at the next meeting.

Art. III. *Duties of Officers.*—The President, or in his absence, the Vice-President, shall preside at meetings of the Institute and of the Board. At the Institute meetings the presiding Officer shall vote only in case of a tie, but at the Board meetings he may vote in all cases.

The Secretary shall keep a full and accurate record of the proceedings at the meetings of the Institute and of the Board, send out calls for the said meetings, and, with the approval of the President, carry on the correspondence of the Institute. Subject to the direction of the Board, he shall have immediate charge of the office, and archives, and other chattels of the Institute.

The Secretary and Librarian, under the general supervision of the Board, shall have charge of all matters connected with editing

and printing the Institute Publications. There shall be a publication known as *The Record*, which shall contain only the proceedings of the meetings, original papers presented by Fellows and written by Fellows or Associates, and such discussions or other matters as may be expressly authorized by the Board of Governors.

The Treasurer shall send out calls for annual dues and acknowledge receipt of same; pay all bills approved by the President for expenditures authorized by the Board or the Institute; keep a detailed account of all receipts and expenditures, and present an abstract of the same at the annual meetings after it has been audited by a Committee of the Board.

Art. IV. *Dues.*—Each Fellow shall pay ten dollars, each Contributing Member ten dollars, each Associate five dollars, upon entrance, and a like amount at each annual meeting thereafter, except in the case of Fellows not residing on this continent, who shall pay five dollars annually at the time stated.

It shall be the duty of the Treasurer to notify by mail any Fellow, Associate, or Contributing Member whose dues may be six months in arrears, and to accompany such notice by a copy of this article. If such Fellow, Associate, or Contributing Member shall fail to pay such dues within three months from the time of mailing such notice, the Treasurer shall report the delinquent to the Board, by whom the name of such member shall be stricken from the rolls, and such member shall cease to be a Fellow, Associate, or Contributing Member of the Institute. Such Member may, however, be reinstated by vote of the Board and upon payment of arrears of dues.

Art. V. *Designation by Initials.*—Fellows of the Institute are authorized to append to their names the initials F.A.I.A., and Associates are authorized to append to their names the initials A.A.I.A.

Art. VI. *Amendments.*—These by-laws may be amended by an affirmative vote of two-thirds of the Fellows present at any meeting held at least three months after notice of the proposed amendment shall have been sent to each Fellow by the Secretary.